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2ND SERIES, VOL. X. NO. 1

OCTOBER, 1903

Whole No. 121

CONTENTS

PAGE

Poincaré's Review of Hilbert's "Foundations of Geometry." Translated by Dr. E. V. HUNTINGTON, 1
On Linear Differential Congruences. By Dr. SAUL EPSTEEN, 23
Fields whose Elements are Linear Differential Expressions. By Professor L. E. DICKSON, 30
On Directrix Curves of Quintic Scrolls. By Mr. C. H. SISAM. 32
Josiah Willard Gibbs, Ph.D. LL.D. A Short Sketch and Appreciation of his Work in Pure Mathematics. By Professor P. F. SMITH, 34
Notes, 39
New Publications, 44

2ND SERIES, VOL. X. NO. 2

NOVEMBER, 1903

Whole No. 122

CONTENTS

PAGE

The Tenth Summer Meeting of the American Mathematical Society. By Professor F. N. COLE, 53
Report of the Committee of the American Mathematical Society on Definitions of College Entrance Requirements, 74
On the Congruence $x^{\phi(P)} \equiv 1$, Mod. P^n. By Professor JACOB WESTLUND, 78
Mach's Mechanics. By Dr. E. B. WILSON, 80
Forsyth's Differential Equations. By Professor E. J. WILCZYNSKI, 86
Notes, 93
New Publications, 104

2ND SERIES, VOL. X. NO. 3

DECEMBER, 1903

Whole No. 123

CONTENTS

	PAGE
The Boston Colloquium of the American Mathematical Society. By Professor F. N. COLE,	119
Linear Systems of Curves upon Algebraic Surfaces. By Professor H. S. WHITE,	120
An Expression of Certain Known Functions as Generalized Hypergeometric Functions. By Mr. E. T. WHITTAKER,	125
On the Factoring of Large Numbers. By Professor F. N. COLE,	134
Note on the p-Discriminant of Ordinary Linear Differential Equations. By Professor ARNOLD EMCH,	137
Hydrodynamic Action at a Distance. By Dr. E. B. WILSON,	139
Shorter Notices,	153
Notes,	158
New Publications,	164

2ND SERIES, VOL. X. NO. 4

JANUARY, 1904

Whole No. 124

CONTENTS

	PAGE
The October Meeting of the American Mathematical Society. By Professor F. N. COLE,	171
Two Systems of Subgroups of the Quaternary Abelian Group in a General Galois Field. By Professor L. E. DICKSON,	178
The Determination of the Constants in the Problem of the Brachistochrone. By Professor OSKAR BOLZA,	185
On Three Types of Surfaces of the Third Order Regarded as Double Surfaces of Translation. By Dr. A. S. GALE,	188
On the Generation of Finite from Infinitesimal Transformations—a Correction. By Professor H. B. NEWSON,	191
Study's Geometry of Dynames. By Professor VIRGIL SNYDER,	193
Encyklopädie der Elementar-Mathematik. By Professor D. E. SMITH,	200
Shorter Notices,	204
Notes,	210
New Publications,	215

2ND SERIES, VOL. X. NO. 5

FEBRUARY, 1904

Whole No. 125

CONTENTS

	PAGE
The Tenth Annual Meeting of the American Mathematical Society. By Professor F. N. COLE,	221
The Cassel Meeting of the Deutsche Mathematiker-Vereinigung. By Mr. R. E. WILSON,	230
On a Test for Non-Uniform Convergence. By Dr. W. H. YOUNG,	239
On the Condition that a Point Transformation of the Plane be a Projective Transformation. By Mr. ELIJAH SWIFT,	247
Note on Cauchy's Integral. By Dr. O. D. KELLOGG,	255
Bauer's Algebra. By Professor L. E. DICKSON,	257
Shorter Notices,	261
Notes,	267
New Publications,	271

2ND SERIES, VOL. X. NO. 6

MARCH, 1904

Whole No. 126

CONTENTS

	PAGE
The December Meeting of the San Francisco Section. By Professor G. A. MILLER,	281
The Fifty-Third Annual Meeting of the American Association for the Advancement of Science. By Professor L. G. WELD,	287
On a Gap in the Ordinary Presentation of Weierstrass's Theory of Functions. By Professor W. F. OSGOOD,	294
On the Theorem of Analysis Situs Relating to the Division of the Plane or of Space by a Closed Curve or Surface. By Mr. L. D. AMES,	301
The Theory of Waves. By Dr. E. B. WILSON,	305
Burkhardt's Theory of Functions. By Professor L. E. DICKSON,	317
Notes,	321
New Publications,	324

2ND SERIES, VOL. X. NO. 7

APRIL, 1904

Whole No. 127

CONTENTS

PAGE

The Winter Meeting of the Chicago Section. By Professor THOMAS F. HOLGATE, 329
On Doubly Infinite Systems of Directly Similar Convex Arches with Common Base Line. By Professor E. H. MOORE, 337
The Riccati Differential Equations which Represent Isothermal Systems. By Dr. EDWARD KASNER, 341
On Some Properties of Groups whose Orders are Powers of a Prime. By Dr. W. B. FITE, 346
Whittaker's Modern Analysis. By Professor MAXIME BÔCHER, 351
Enriques's Projective Geometry. By Professor VIRGIL SNYDER, 355
Shorter Notices, 359
Acknowledgment, 361
Errata, 362
Notes, 362
New Publications, 367

2ND SERIES, VOL. X. NO. 8

MAY, 1904

Whole No. 128

CONTENTS

PAGE

The February Meeting of the American Mathematical Society. By Professor F. N. COLE, 373
Report on the Requirements for the Master's Degree. By the Committee of the Chicago Section, 380
On the Subgroups of Order a Power of p in the Linear Homogeneous and Fractional Groups in the $GF[p^n]$. By Professor L. E. DICKSON, 385
The Exterior and Interior of a Plane Curve. By Dr. G. A. BLISS, 398
Riccati Isothermal Systems—A Correction. By Dr. EDWARD KASNER, 405
Shorter Notices, 406
Notes, 413
New Publications, 424

2ND SERIES, VOL. X. NO. 9

JUNE, 1904

Whole No. 129

CONTENTS

	PAGE
The April Meeting of the Chicago Section. By Professor THOMAS F. HOLGATE,	429
The Heine-Borel Theorem. By Dr. OSWALD VEBLEN,	436
On Self-Dual Scrolls. By Professor C. H. SISAM,	440
On Some Tendencies in Geometric Investigations. By Professor CORRADO SEGRE,	442
Reply to Professor Snyder's Review of Study's Geometrie der Dynamen. By Professor E. STUDY,	468
Notes,	471
New Publications,	479

2ND SERIES, VOL. X. NO. 10

JULY, 1904

Whole No. 130

CONTENTS

	PAGE
The April Meeting of the American Mathematical Society. By Professor F. N. COLE,	485
The April Meeting of the San Francisco Section. By Professor G. A. MILLER,	494
On Linear Homogeneous Difference Equations and Continuous Groups. By Dr. SAUL EPSTEEN,	499
Experimental and Theoretical Geometry. By Professor R. E. MORITZ,	504
Shorter Notices,	511
Notes,	518
New Publications,	524
Thirteenth Annual List of Published Papers,	530
Index of Volume,	537

BULLETIN OF THE

AMERICAN MATHEMATICAL SOCIETY.

POINCARÉ'S REVIEW OF HILBERT'S "FOUNDATIONS OF GEOMETRY."*

What are the fundamental principles of geometry? what is its origin? its nature? its scope? These are questions which have at all times engaged the attention of mathematicians and thinkers, but which about a century ago took on an entirely new aspect, thanks to the ideas of Lobachevsky and of Bolyai.

For a long time we attempted to demonstrate the proposition known as the *postulate of Euclid;* we constantly failed; we know now the reason for these failures. Lobachevsky succeeded in building a logical edifice as coherent as the geometry of Euclid, but in which the famous postulate is assumed false, and in which the sum of the angles of a triangle is always less than two right angles. Riemann devised another logical system, equally free from contradiction, in which this sum is on the other hand always greater than two right angles. These two geometries, that of Lobachevsky and that of Riemann, are what are called the *non-euclidean geometries.* The postulate

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* Translated for the BULLETIN, with the author's permission, by Dr. E. V. HUNTINGTON. The original review appeared in Darboux's *Bulletin des Sciences Mathématiques*, 2d ser., vol. 26 (September, 1902), pp. 249–272, and also, with some modification of the more technical passages, in the *Journal des Savants* for 1902 (May), pp. 252–271. The present translation (except the postscript) is from the version in Darboux's *Bulletin*, the heavy faced figures in brackets indicating the pages of the original; the postscript appeared only in the *Journal des Savants* (p. 271).

The Grundlagen der Geometrie, by Professor David Hilbert (Leipzig, Teubner, 8vo, 92 pp.), appeared in 1899, and was reviewed for the BULLETIN by Dr. Sommer (vol. 6, 1900, pp. 287–299). A French translation by Professor L. Laugel and an English translation by Professor E. J. Townsend appeared in 1900 and 1902, and were reviewed by Dr. E. R. Hedrick in the BULLETIN, vol. 9 (1902), pp. 158–165. See also a review by Mr. O. Veblen in the *Monist*, vol. 13 (January, 1903), pp. 303–309.

of Euclid then cannot be demonstrated; and this impossibility is as absolutely certain as any mathematical truth whatsoever — a fact which does not prevent the Académie des Sciences from receiving every year several new proofs, to which it naturally refuses the hospitality of the *Comptes rendus*.

Much has already been written on the non-euclidean geometries; once they scandalized us; now we have become accustomed to their paradoxes; some people have gone so far as to doubt the truth of the postulate and to ask whether real space is plane, as Euclid assumed, or whether it may not present a slight *curvature*. They even supposed that experiment could give them an answer [250] to this question. Needless to add that this was a total misconception of the nature of geometry, which is not an experimental science.

But why, among all the axioms of geometry, should this postulate be the only one which could be denied without offence to logic? Whence should it derive this privilege? There seems to be no good reason for this, and many other conceptions are possible.

However, many contemporary geometers do not appear to think so. In recognizing the claims of the two new geometries they feel doubtless that they have gone to the extreme limit of possible concessions. It is for this reason that they have conceived what they call *general geometry*, which includes as special cases the three systems of Euclid, Lobachevsky, and Riemann, and does not include any other. And this term *general* indicates clearly that, in their minds, no other geometry is conceivable.

They will lose this illusion if they read the work of Professor Hilbert. In it they will find the barriers behind which they have wished to confine us broken down at every point.

To understand well this new attempt we must recall what has been the evolution of mathematical thought for the last hundred years, not only in geometry, but in arithmetic and in analysis. The concept of number has been made more clear and precise; at the same time it has been generalized in various directions. The most valuable of these generalizations for the analyst is the introduction of *imaginaries* which the modern mathematician could not now dispense with; but we have not stopped with this; other generalizations of number, or, as we say, other categories of complex numbers, have been introduced into science.

The operations of arithmetic have in their turn been subjected to criticism, and Hamilton's quaternions have given us an example of an operation which presents an almost perfect analogy to multiplication, and may be called by the same name, which, however, is not commutative, that is, the product of two factors is not the same when the order of the factors is reversed. This was a revolution in arithmetic quite comparable to that which Lobachevsky effected in geometry.

Our conception of the infinite has been likewise modified [251]. Professor G. Cantor has taught us to distinguish gradations in infinity itself (which have, however, nothing to do with the infinitesimals of different orders invented by Leibniz for the ordinary infinitesimal calculus). The concept of the continuum, long regarded as a primitive concept, has been analyzed and reduced to its elements.

Shall I mention also the work of the Italians, who have endeavored to construct a universal logical symbolism and to reduce mathematical reasoning to purely mechanical rules?

We must recall all this if we wish to understand how it is possible that conceptions which would have staggered Lobachevsky himself, revolutionary as he was, can seem to us to-day almost natural, and can be propounded by Professor Hilbert with perfect equanimity.

THE LIST OF AXIOMS.—The first thing to do was to enumerate all the axioms of geometry. This was not so easy as one might suppose; there are the axioms which one sees and those which one does not see, which are introduced unconsciously and without being noticed. Euclid himself, whom we suppose an impeccable logician, frequently applies axioms which he does not expressly state.

Is the list of Professor Hilbert final? We may take it to be so, for it seems to have been drawn up with care. The distinguished professor divides the axioms into five groups:

I. Axiome der Verknüpfung (I shall translate by *projective axioms* [*axiomes projectifs*] instead of trying to find a literal translation, as for example *axioms of connection* [*axiomes de la connection*], which would not be satisfactory).

II. Axiome der Anordnung (axioms of order [*axiomes de l'ordre*]).

III. Axiom of Euclid.

IV. Axioms of congruence or metrical axioms.

V. Axiom of Archimedes.

Among the projective axioms, we shall distinguish those of the plane and those of space; the first are those derived from the familiar proposition: *through two points passes one and only one straight line;* — but I prefer to translate literally, in order to make Professor Hilbert's thought well understood.

"Let us suppose three systems of objects which we shall call *points*, [252] *straight lines*, and *planes*. Let us suppose that these points, straight lines, and planes are connected by certain relations which we shall express by the words *lying on*, *between*, etc.

"I. — 1. Two different points A and B determine always a straight line a; in notation

$$AB = a \text{ or } BA = a.$$

"In place of the word *determine* we shall employ as well other turns of phrase which shall be synonymous; we shall say: A lies on a, A is a point of a, a passes through A, a joins A and B, etc.

"I. — 2. Any two points of a straight line determine this straight line; that is, if $AB = a$ and $AC = a$, and if B is different from C, we have also $BC = a$."

The following are the considerations which these statements are intended to suggest: the expressions *lying on*, *passing through*, etc., are not meant to call up mental pictures; they are simply synonyms of the word *determine*. The words *point*, *straight line*, and *plane* themselves are not intended to arouse in the mind any visual image [*représentation sensible*]. They might denote indifferently objects of any sort whatever, provided one could establish among these objects a correspondence such that to every pair of the objects called *points* there would correspond one and only one of the objects called *straight lines*. And this is why it becomes necessary to add (I, 2) that, if the line which corresponds to the pair of points A and B is the same as that which corresponds to the pair of points B and C, it is also the same as that which corresponds to the pair of points A and C.

Thus Professor Hilbert has, so to speak, sought to put the axioms into such a form that they might be applied by a person who would not understand their meaning because he had never seen either point or straight line or plane. It should be possible, according to him, to reduce reasoning to purely me-

chanical rules, and it should suffice, in order to create geometry, to apply these rules slavishly to the axioms without knowing what the axioms mean. We shall thus be able to construct all geometry, I will not say precisely without understanding it at all, since we shall grasp the logical connection of the [253] propositions, but at any rate without seeing it at all. We might put the axioms into a reasoning apparatus like the logical machine* of Stanley Jevons, and see all geometry come out of it.

This is the same consideration that has inspired certain Italian scholars, such as Peano and Padoa, who have endeavored to create a *pasigraphy*, that is, a sort of universal algebra, where all the processes of reasoning are replaced by symbols or formulas.

This notion may seem artificial and puerile; and it is needless to point out how disastrous it would be in teaching and how hurtful to mental development; how deadening it would be for investigators, whose originality it would nip in the bud. But, as used by Professor Hilbert, it explains and justifies itself, if one remembers the end pursued. Is the list of axioms complete, or have we overlooked some which we apply unconsciously? This is what we want to know. For this we have one criterion, and only one. We must find out whether geometry is a logical consequence of the axioms explicitly stated, that is, whether, if we put these axioms into the reasoning machine, we can make the whole sequence of propositions come out.

If we can, we shall be sure that nothing has been overlooked. For our machine cannot work except according to the rules of logic for which it has been constructed; it ignores the vague instinct which we call *intuition*.

I shall not enlarge upon the projective axioms of space, which the author numbers I, 3, 4, 5, 6. Nothing is changed from the usual statements.

A word only on the axiom I, 7, which is thus formulated:

"On every straight line there are at least two points; on every plane there are at least three points not in a straight line; in space there are at least four points which are not in the same plane."

This statement is characteristic. Any one who had left any place for intuition, however small it might be, would not have

* [*Cf. Lond. Phil. Trans.*, vol. 160 (1870), pp. 497–518. *Tr.*]

dreamed of saying that on every straight line there are at least two points, or rather he would have added at once that there are an infinite number of them; for the intuition of the [254] straight line would have revealed to him both facts immediately and simultaneously.

Let us pass to the second group, that of the axioms of order. Here is the statement of the first two:

"If three points are on the same straight line, there is a certain relation among them which we express by saying that one of the points, and only one, is between the other two. If C is between A and B, and D between A and C, then D will be also between A and B, etc."

Here again we do not bring in our intuition; we are not seeking to fathom what the word *between* may signify; every relation which satisfies the axioms might be denoted by the same word. This is an illuminating example of the purely formal nature of mathematical definitions; but I do not dwell upon it, since I should have simply to repeat what I have said already, in speaking of the first group.

But another consideration forces itself upon us. The axioms of order are presented as dependent on the projective axioms, and they would not have any meaning if we did not admit these latter, since we should not know what are three points on a straight line. And nevertheless there exists a special geometry, purely qualitative, which is entirely independent of projective geometry, and does not assume the idea of the straight line, nor that of the plane, but only the ideas of curves and surfaces; this is what is called *analysis situs*. Would it not be preferable to give to the axioms of the second group a form which would free them from this dependence and separate them completely from the first group? It remains to be seen whether this would be possible, while preserving the purely logical character of these axioms, that is, while closing the door completely against all intuition.

The third group contains only a single axiom, which is the famous postulate of Euclid; I shall note simply that, contrary to the usual custom, it is presented before the metrical axioms.

These last form the fourth group. We shall divide them into three subgroups. The propositions IV, 1, 2, 3 are the metrical axioms for segments: these axioms serve to define length. We shall agree to say that a segment taken on a [255] straight line may be congruent (equal) to a segment taken on

another straight line; this is axiom IV, 1; but this convention is not wholly arbitrary; it must be so made that two segments congruent to the same third segment shall be congruent to each other (IV, 2). In the next place we define the addition of segments, by a new convention; and this convention, in turn, must be so made that when we add equal segments we find the sums equal; and this is axiom IV, 3.

The propositions IV, 4; 5 are the corresponding axioms for angles. But these are not yet sufficient; to the two subgroups of metrical axioms for segments and for angles we must add the metrical axiom for triangles (which Professor Hilbert numbers IV, 6): if two triangles have an equal angle included between equal sides, the other angles of these two triangles are equal each to each.

We recognize here one of the well known cases of equality of triangles, which we usually demonstrate by superposition, but which we must set up as a postulate if we wish to avoid making appeal to intuition. Moreover, when we made use of intuition, that is of superposition, we saw by the same process that the third sides were equal in the two triangles, and these two propositions were united, so to speak, in a single apperception; here, on the contrary, we separate them; one of them we make a postulate, but we do not set up the other as a postulate, since it can be logically deduced from the first.

Another comment: Professor Hilbert says distinctly that the segment AB is congruent to itself, but (and the same is true for angles) he should have added, should he not, that it is congruent to the inverse segment BA. This axiom (which implies the symmetry of space) is not identical with those which are explicitly stated. I do not know whether it could be logically deduced from them; I believe it could, but, given the course of reasoning of Professor Hilbert, it seems to me that this postulate is applied without being stated (page 17, line 18).

I also regret that, in this exposition of the metrical axioms, there remains no trace of an idea whose importance Helmholtz was the first to understand: I refer to the displacement of a rigid figure. It would have been possible to preserve this idea in its natural rôle, without sacrificing the logical character of the axioms. One might have said, for example: I define between figures a [256] certain relation which I call *congruence*, etc.; two figures which are congruent to the same third figure are congruent to each other; two congruent figures are

identical when three points of one, not in a straight line, are identical with three corresponding points of the other, etc. The artificial introduction of this axiom IV, 6 would thus have been avoided, and the postulates would have been brought into connection with their actual psychological origin.

The fifth group contains only a single axiom, that of Archimedes.

Let A and B be any two points on a straight line D; let a be any segment; starting from the point A, and in the direction AB, construct on D a series of segments, all equal to each other and equal to a: AA_1, A_1A_2, $\cdots$, $A_{n-1}A_n$; then we shall always be able to take n so great that the point B will be found on one of these segments.

That is to say, if we have given any two lengths l and L, we can always find a whole number n so great that when we add the length l to itself n times, we obtain a total length greater than L.

INDEPENDENCE OF THE AXIOMS. — The list of axioms once drawn up, we must see whether it is free from contradiction. We know well that it is, since geometry exists; and Professor Hilbert also answers in the affirmative, by constructing a geometry. But this geometry, strange to say, is not quite the same as ours, his space is not our space, or at least is only a part of it. In the space of Professor Hilbert we do not have all the points which there are in our space, but only those which we can construct by ruler and compass, starting from two given points. In this space, for example, there would not exist, in general, an angle which would be the third part of a given angle.

I have no doubt that this conception would have been regarded by Euclid as more rational than ours. At any rate it is not ours. To come back to our geometry it would be necessary to add an axiom:

"If, on a straight line, there is a double infinity of points $A_1, A_2, \cdots, A_n, \cdots$; $B_1, B_2, \cdots, B_n, \cdots$, such that B_q is included between A_p and [257] B_{q-1}, and A_p between B_q and A_{p-1}, whatever the values of p and q, then there will be on this straight line at least one point C which lies between A_p and B_q, whatever the values of p and q."

We must ask next whether the axioms are independent, that is, whether we could sacrifice one of the five groups, retaining

the other four, and still attain a coherent geometry. Thus by suppressing group III (the postulate of Euclid), we obtain the non-euclidean geometry of Lobachevsky. In the same way, we can suppress group IV.

Professor Hilbert has succeeded in retaining groups I, II, III and V, along with the two subgroups of metrical axioms for segments and for angles, while rejecting the metrical axiom for triangles, that is, proposition IV, 6.

This is how he accomplishes it: consider, for simplicity, plane geometry, and let P be the plane in which we operate; we shall retain the usual meaning for the words *point* and *straight line*, and also the usual measurement of angles; but not so for lengths. A length shall be measured *by definition* by its projection on a plane Q different from P, this projection itself being measured in the usual way. It is clear that all the axioms will hold, except the metrical axioms. The metrical axioms for angles will also hold, since we change nothing concerning the measurement of angles; those for segments will also hold, since each segment is measured by another segment which is its projection on the plane Q, and this latter segment is measured in the usual way. On the other hand, the theorems on the equality of triangles, such as the axiom IV, 6, are no longer true. This solution satisfies me only half-way; angles have been defined independently of lengths, without trying to bring the two definitions into agreement (or rather, by bringing them purposely into disagreement). To return to classic geometry it would be sufficient to change *one* of the two definitions. I should prefer to have had the lengths so defined as to make it impossible to find a definition of angles satisfying the metrical axioms for angles and for triangles. This would moreover not be difficult [258].

It would have been easy for Professor Hilbert to create a geometry in which the axioms of order would be abandoned while all the others would be retained. Or rather this geometry exists already, or rather there exist two of them. There is that of Riemann, for which, it is true, the postulate of Euclid (group III) is also abandoned, since the sum of the angles of a triangle is greater than two right angles. To make my thought clear I shall limit myself to considering a geometry of two dimensions. The geometry of Riemann in two dimensions is nothing else than spherical geometry, with one condition,

namely, that we shall not regard as distinct two diametrically opposite points on the sphere. The elements of this geometry will then be the different diameters of this sphere. Now, if we consider three diameters of the same sphere, lying in the same diametral plane, we have no reason for saying that one of them is *between* the other two. The word *between* has no longer any meaning, and the axioms of order drop out of themselves.

If we wish now a geometry in which the axioms of order shall not hold, while the axiom of Euclid is retained with the others, we have only to take as elements the *imaginary* points and straight lines in ordinary space. It is clear that the imaginary points of space are not given us as *arranged* in a definite order. But more than that: we may ask whether they are capable of being so arranged; this would undoubtedly be possible, as G. Cantor has shown (subject to the condition, be it understood, of not always arranging in close proximity points which we regard as infinitely near, and of destroying thereby the continuity of space). We might, I say, arrange them, but this could not be done in such a way that the arrangement would not be altered by the various operations of geometry (projection, translation, rotation, etc.). The axioms of order, then, are not applicable to this geometry.

THE NON-ARCHIMEDEAN GEOMETRY. — But the most original conception of Professor Hilbert is that of non-archimedean geometry, in which all the axioms remain true except that of Archimedes. For this it was necessary, in the first place, to construct a *system* [259] *of non-archimedean numbers*, that is, a system of elements among which we may define the relations of equality and inequality and to which we may apply operations analogous to arithmetical addition and multiplication — and this in such a way as to satisfy the following conditions:

1° The arithmetical rules for addition and multiplication (the commutative, associative, distributive laws, etc.: *Arithmetische Axiome der Verknüpfung*) hold without change.

2° The rules for the establishment and transformation of inequalities (*Arithmetische Axiome der Anordnung*) likewise hold.

3° The axiom of Archimedes is not true.

We may reach this result by choosing for elements series of the following form:

$$A_0 t^m + A_1 t^{m-1} + A_2 t^{m-2} + \cdots,$$

where m is a positive or negative integer and where the coefficients A are real, and by agreeing to apply to these series the ordinary rules of addition and multiplication. We must then define the conditions of inequality of these series, so as to *arrange* our elements in a definite order. We shall accomplish this by the following convention: we shall give to our series the sign of A_0 and we shall say that one series is less than another when, subtracted from the other, it leaves a positive remainder.

It is clear that with this convention the rules of the calculus of inequalities hold; but the axiom of Archimedes is no longer true; for, if we take the two elements 1 and t, the first added to itself as many times as we please remains always less than the second. We shall have always $t > n$, whatever the whole number n, since the difference $t - n$ will always be positive; for the coefficient of the first term t, which, by definition, gives its sign, remains always equal to 1.

Our ordinary numbers come in as particular cases among these *non-archimedean numbers*. The new numbers are interpolated, so to speak, in the series of our ordinary numbers, in such a way that we may have, for example, an infinity of the new numbers less than a given ordinary number A and greater than all the ordinary numbers less than A [260].

This premised, imagine a space of three dimensions in which the coordinates of a point would be measured not by ordinary numbers but by non-archimedean numbers, while the usual equations of the straight line and the plane would hold, as well as the analytic expressions for angles and lengths. It is clear that in this space all the axioms would remain true except that of Archimedes.

On every straight line new points would be interpolated between our ordinary points. If, for example, D_0 is an ordinary straight line, and D_1 the corresponding non-archimedean straight line; if P is any ordinary point of D_0, and if this point divides D_0 into two half-rays S and S′ (I add, for precision, that I consider P as not belonging to either S or S′); then there will be on D_1 an infinity of new points as well between P and S as between P and S′. There will be also on D_1 an infinity of new points which will lie to the right of all the ordinary points of D_0. In short, our ordinary space is only a part of the non-archimedean space.

At the first blush the mind revolts against conceptions like

this. This is because, through an old habit, it is looking for a visual image. It must free itself from this prejudice if it would arrive at comprehension, and this is even more necessary here than in the case of non-euclidean geometry. Professor Hilbert has only one object in view: to construct a system of elements capable of certain logical relations; and it is sufficient for him to show that these relations do not involve any self-contradiction.

We may remark in passing that the non-euclidean geometry respects, so to speak, our qualitative conception of the geometrical continuum, while entirely overturning our ideas about the measurement of this continuum. The non-archimedean geometry destroys this concept, by dissecting the continuum for the introduction of new elements.

Whatever they may be, Professor Hilbert follows out the consequences of his premises and tries to see how one could remake geometry without using the axiom of Archimedes. There is no difficulty in the chapters which the school-boys call the *first* and *second* Books. This axiom does not occur at any point in those Books.

The third Book treats of proportions and of similarity. The plan which Professor Hilbert follows for the [261] reconstruction of this book without recourse to the axiom of Archimedes is, in substance, as follows. He takes the usual construction of the fourth proportional as the definition of proportion; but such a definition needs to be justified; he needs to show in the first place that the result is the same whatever may be the auxiliary lines employed in the construction, and next that the ordinary rules of operation apply to the proportions thus defined. This justification Professor Hilbert gives us in a satisfactory manner.

The fourth Book treats of the measurement of plane areas. If this measurement can be easily established without the aid of the principle of Archimedes, it is because two equivalent polygons can either be decomposed into triangles in such a way that the component triangles of the one and those of the other are equal each to each (or, in other words, can be converted one into the other after the manner of the Chinese puzzle *), or else can be regarded as the difference of polygons capable of this mode of decomposition (this is really the same process, admit-

* [By cutting up and putting together again. *T*.]

ting not only positive triangles but also negative triangles). But we must observe that an analogous state of affairs does not seem to exist in the case of two equivalent polyhedra, so that it becomes a question whether we can determine the volume of the pyramid, for example, without an appeal more or less disguised to the infinitesimal calculus. It is then not certain whether we could dispense with the axiom of Archimedes as easily in the measurement of volumes as in that of plane areas. Moreover Professor Hilbert has not attempted it.

One question remains to be treated in any case; a polygon being given, is it possible to cut it up into triangles and remove one of the pieces in such a way that the remaining polygon may be equivalent to the given polygon, that is to say, in such a way that by transforming this remaining polygon by the process of the Chinese puzzle we could come back to the original polygon? Ordinarily we are satisfied with saying that this is impossible because the whole is greater than the part. This is to call in a new axiom, and, however obvious it may seem to us, the logician would be better satisfied if we could avoid it. Professor Schur has discovered a proof, it is true, but it depends on the axiom of Archimedes; Professor Hilbert wished to reach the result without using this axiom. This is the device by which he [262] does it: he adopts as the *definition* of the *area* of the triangle half the product of its base by its altitude, and he justifies this definition by showing that two triangles which are equivalent (from the point of view of the Chinese puzzle) have the same *area* (in the sense of the new definition) and that the *area* of a triangle which can be decomposed into several others is the sum of the *areas* of the component triangles. This justification once out of the way, all the rest follows without difficulty. It is always the same process. To avoid constant appeals to intuition, which would provide us constantly with new axioms, we change these axioms into definitions, and afterwards justify these definitions by showing that they are free from contradictions.

THE NON-ARGUESIAN GEOMETRY. — The fundamental theorem of projective geometry is the theorem of Desargues. Two triangles are called *homologous* when the straight lines which join the corresponding vertices intersect in the same point. Desargues has shown that the points of intersection of the corresponding sides of two homologous triangles are on the same straight line; the converse is also true.

The theorem of Desargues can be established in two ways:

1° By using the projective axioms of the plane and the metrical axioms of the plane.

2° By using the projective axioms of the plane and those of space.

The theorem might then be discovered by a two-dimensional animal, to whom a third dimension would seem as inconceivable as a fourth does to us; such an animal would then be ignorant of the projective axioms of space; but he would have seen movement, in the plane which he inhabits, of rigid figures analogous to our rigid bodies, and would consequently be acquainted with the metrical axioms. The theorem could be discovered also by a three-dimensional animal who was acquainted with the projective axioms of space, but who, never having seen rigid bodies move, would be ignorant of the metrical axioms.

But could we establish the theorem of Desargues without using either the projective axioms of space or the metrical axioms, [263] but only the projective axioms of the plane? We thought not, but we were not sure of it. Professor Hilbert has decided the question by constructing a *non-arguesian geometry*, which is, of course, a plane geometry. Consider an ellipse E. Outside of this ellipse the word *straight line* preserves its ordinary meaning: in the interior the word *straight line* takes a different meaning and denotes an arc of a circle which, when produced, would pass through a fixed point P outside the ellipse. A straight line which crosses the ellipse E is then composed of two rectilinear parts, in the ordinary sense of the word, connected in the interior of the ellipse by an arc of a circle; like a ray of light which would be deflected from its rectilinear path by passing through a refracting body.

The projective axioms of the plane will still be true if we take the point P sufficiently far removed from the ellipse E.

Now place two homologous triangles outside the ellipse E, and in such a way that their sides do not meet E; the three straight lines which join the corresponding vertices two and two, *if we take them in the ordinary sense of the word*, will meet in the same point Q, according to the theorem of Desargues; suppose that this point Q is in the interior of E. *If we take the word* straight line *in the new sense*, the three straight lines which join the corresponding vertices will be deflected on entering the interior of the ellipse. They will then no longer pass

through Q, they will be no longer concurrent. The theorem of Desargues is no longer true in our new geometry; this is a non-arguesian geometry.

THE NON-PASCALIAN GEOMETRY. — Professor Hilbert does not stop here, but introduces still another new conception. In order to understand it, we must first return a moment into the domain of arithmetic. We have noticed above the extension of the concept of number, by the introduction of the *non-archimedean numbers*. We want a classification of these new numbers, to obtain which we shall begin by classifying the axioms of arithmetic in four groups, which are:

1° The associative and commutative laws of addition, the associative law of multiplication, the two [264] distributive laws of multiplication; or, in short, all the rules of addition and of multiplication, except the commutative law of multiplication;

2° The axioms of order; that is, the rules of the calculus of inequalities;

3° The commutative law of multiplication, according to which we can invert the order of the factors without changing the product;

4° The axiom of Archimedes.

Numbers which admit the axioms of the first two groups shall be called *arguesian;* they may be *pascalian* or *non-pascalian*, according as they satisfy or do not satisfy the axiom of the third group; they will be *archimedean* or *non-archimedean*, according as they satisfy or not the axiom of the fourth group. We shall soon see the reason for these names.

Ordinary numbers are at the same time arguesian, pascalian and archimedean. It can be shown that the commutative law follows from the axioms of the first two groups and the axiom of Archimedes; there are therefore no numbers which are arguesian, archimedean and not pascalian.

On the other hand, we have cited above an example of numbers which were arguesian, pascalian and not archimedean; I shall call these the *numbers of the system* T, and I recall that to each of these numbers there corresponds a series of the form

$$A_0 t^m + A_1 t^{m-1} + \cdots,$$

where the A's are ordinary real numbers.

It is easy to construct, by an analogous process, a system of arguesian numbers which are non-pascalian and non-archimedean. The elements of this system will be series of the form

$$S = T_0 s^n + T_1 s^{n-1} + \cdots,$$

where s is a symbol analogous to t, n a positive or negative integer, and $T_0, T_1, \cdots$ numbers of the system T; if then we replaced the coefficients $T_0, T_1, \cdots$ by the corresponding series in t we should have a series depending on both t and s. We shall [265] add these series S according to the ordinary rules; also for the multiplication of these series we shall admit the distributive and associative laws; but we shall suppose that the commutative law is not true, that on the contrary $st = -ts$.

It remains to *arrange* the series in a definite order, so as to satisfy the axioms of order. For this, we shall attribute to the series S the sign of the first coefficient T_0; we shall say that one series is less than another when, subtracted from the first, it leaves a positive remainder. It is always the same scheme: t is regarded as very great in comparison with any ordinary real number, and s is regarded as very great in comparison with any number of the system T.

The commutative law not being true, these are clearly non-pascalian numbers.

Before going farther, I recall that Hamilton introduced long ago a system of complex numbers in which multiplication is not commutative; these are the *quaternions*, of which the English make such frequent use in mathematical physics. But, in the case of quaternions, the axioms of order are not true; the originality of Professor Hilbert's conception lies in this, that his new numbers satisfy the axioms of order without satisfying the commutative law.

To return to geometry. Admit the axioms of the first three groups, that is, the projective axioms of the plane and of space, the axioms of order and the postulate of Euclid: the theorem of Desargues will follow from them, since it is a consequence of the projective axioms of space.

We wish to construct our geometry *without making use of the metrical axioms;* the word *length* has then for us no meaning; we have no right to use the compass; on the other hand, we may use the ruler, since we admit that we can draw a straight line through two points, by virtue of one of the projective axioms;

also, we know how to draw through a given point a parallel to a given straight line, since we admit the postulate of Euclid. Let us see what we can do with these resources.

We can define the *homothetic relation* * [*homothétie*] of two figures; two triangles shall be called homothetic when their sides are [266] parallel two and two, and we conclude from this (by the theorem of Desargues, which we admit) that the straight lines which join the corresponding vertices are concurrent. We shall then make use of the homothetic relation to define proportion. We can also define equality to a certain extent.

Two opposite sides of a parallelogram shall be equal *by definition;* we can thus decide whether two segments are equal to each other, provided they are parallel.

Thanks to these conventions, we are now in a position to compare the lengths of two segments; *provided,* however, *that these segments are* PARALLEL. The comparison of two lengths which have different directions has no meaning; there would be required, so to speak, a different unit of length for each direction. Needless to add that the word *angle* has no meaning.

Lengths will thus be expressed by numbers; but these will not necessarily be ordinary numbers. All that we can say is this, that, if the theorem of Desargues is true, as we admit, these numbers will belong to a system satisfying the arithmetic axioms of the first two groups, that is, to an *arguesian system.* Conversely, being given any system S of arguesian numbers, we can construct a geometry in which the lengths of segments of a straight line can be exactly expressed by these numbers.

Here is the way in which this can be done: a point of this new space shall be *defined* by three numbers x, y, z of the system S which we shall call the *coordinates* of this point. If to the three coordinates of the various points of a figure we add three constants (which are, of course, arguesian numbers of the system S), we obtain another figure, derived from the first in such a way that to any segment of one of the figures there corresponds an equal and parallel segment in the other (in the sense given above to this word). This transformation is then a translation, so that these three constants might define a translation. If now we multiply the three coordinates of all the points of a given figure by the same constant, we shall obtain a second figure which will be homothetic to the first [267].

* [Two figures are *homothetic* when they are similar and similarly placed. *Tr.*]

The equation of a plane will be the well known linear equation of ordinary analytic geometry; but, since in the system S multiplication will not in general be commutative, it is important to make a distinction and to say that in each of the terms of this linear equation that factor shall be the coordinate which plays the rôle of multiplicand, and that the constant coefficient which plays the rôle of multiplier.

Thus, to each system of arguesian numbers there will correspond a new geometry satisfying the projective axioms, the axioms of order, the theorem of Desargues, and the postulate of Euclid. What is now the geometric meaning of the arithmetic axiom of the third group, that is, of the commutative law of multiplication? *Translated into geometric language, this law is the theorem of Pascal;* I refer to the theorem on the hexagon inscribed in a conic, supposing that this conic reduces to two straight lines.

Thus the theorem of Pascal will be true or false according as the system S is pascalian or non-pascalian; and, since there are non-pascalian systems, *there are also non-pascalian geometries.*

The theorem of Pascal can be proved by starting with the metrical axioms; it will then be true, if we admit that figures can be transformed not only by the homothetical transformation and translation, as we have just been doing, but also by rotation.

The theorem of Pascal can also be deduced from the axiom of Archimedes, since we have just seen that every system of numbers which is arguesian and archimedean is at the same time pascalian; *every non-pascalian geometry is then at the same time non-archimedean.*

The Streckenübertrager. — Let us mention one more idea of Professor Hilbert. He studies the constructions which can be made, not with the aid of the ruler and compass, but by means of the ruler and a special instrument which he calls *Streckenübertrager,* and which would enable us to lay off on a straight line a segment equal to another segment taken on another straight line. The *Streckenübertrager* is not equivalent to the compasses; this latter instrument would enable us to construct the point of intersection of two circles or of a [268] circle and any straight line; the *Streckenübertrager* would give us only the intersection of a circle with a straight line *passing*

through the center of the circle. Professor Hilbert inquires then what constructions will be possible with these two instruments, and he reaches a quite remarkable conclusion.

The constructions which can be made by the ruler and compass can also be made by the ruler and the *Streckenübertrager, if these constructions are such that their result is always real.* It is easy enough to see that this condition is necessary; for a circle is always cut in two real points by a straight line drawn through its centre. But it was difficult to foresee that this condition would be also sufficient.

Various Geometries. — I should like, before closing, to see what places are taken in Professor Hilbert's classification by the various geometries which have been proposed up to the present time. In the first place, the geometries of Riemann; I do not mean *the* geometry of Riemann which has been mentioned above and which is contrasted with that of Lobachevsky; I mean the geometries connected with space of variable curvature considered by Riemann in his celebrated *Habilitationsschrift.*

In this conception, any curve has a length assigned to it, by definition, and it is on this definition that everything depends. The rôle of straight lines is played by the geodesics, that is, by the lines of minimum length drawn from one point to another. The projective axioms are no longer true, and there is no reason why, for example, two points could not be joined by more than one geodesic. The postulate of Euclid clearly can no longer have any meaning. The axiom of Archimedes remains true, as well as the axioms of order, *mutatis mutandis;* Riemann does not consider, indeed, any but the ordinary system of numbers. As to the metrical axioms, it is easily seen that those for segments and those for angles remain true, while the metrical axiom for triangles (IV, 6) is evidently false.

And here we meet the objection which has been most often made to Riemann.

"You speak of length," they say to him; "now length assumes measurement, and to measure we must be able to carry about a measuring [269] instrument which must remain invariant; moreover, you recognize this yourself. Space then must be everywhere equal to itself, it must be homogeneous in order that congruence may be possible. Now your space is not homogeneous, since its curvature is variable; in such a space there can be no such thing as measurement or length."

Riemann would have had no trouble in replying. Consider, for simplicity, a geometry of two dimensions; we shall be able then to picture to ourselves Riemann's space as a surface in ordinary space. We might measure lengths on this surface by means of a thread, and nevertheless a figure could not be moved about in this surface in such a way that the lengths of all its elements remain invariant. For the surface is not, in general, applicable on itself.

This is what Professor Hilbert would express by saying that the metrical axioms for segments are true, while that for triangles is not. The first find concrete expression, so to speak, in our thread; the axiom for triangles would assume a displacement of a figure all of whose elements would have a constant length.

What will be the place of another geometry which I have proposed on a former occasion * and which belongs, so to speak, to the same family as that of Lobachevsky and that of Riemann? I have shown that we can imagine three geometries in two dimensions, which correspond respectively to three kinds of surfaces of the second degree: the ellipsoid, the hyperboloid of two sheets, and the hyperboloid of one sheet; the first is that of Riemann, the second is that of Lobachevsky, and the third is the new geometry. We should find in the same way four geometries in three dimensions.

Where would this new geometry stand in the classification of Professor Hilbert? It is easy to discover. As in the case of the geometry of Riemann, all the axioms hold, save those of order and that of Euclid; but, while in the geometry of Riemann the axioms are false on all the straight lines, in the new geometry, on the contrary, the straight lines separate themselves into two classes, those on which the axioms of order are true, and those on which they are false [270].

Conclusions. — But the most important thing is to arrive at a clear understanding of the place which the new conceptions of Professor Hilbert occupy in the history of our ideas on the philosophy of mathematics.

After a first period of naïve confidence in which we cherished the hope of demonstrating everything, came Lobachevsky, the inventor of the non-euclidean geometries.

* [See *Bull. de la Société mathématique de France*, vol. 15 (1887), pp. 203–216. Other articles by Poincaré on the foundations of geometry have appeared in the *Revue de Métaphysique*, vol. 7, and the *Monist*, vol. 9. *Tr.*]

But the true meaning of this discovery was not fathomed all at once; Helmholtz showed in the first place that the propositions of euclidean geometry were no other than the laws of motion of rigid bodies, while the propositions of the other geometries were the laws which might govern other bodies analogous to the rigid bodies — bodies which doubtless do not exist, but whose existence might be conceived without leading to the least contradiction, bodies which we might fabricate if we wished. These laws could not, however, be regarded as experimental, since the solids of nature follow them only roughly, and since, besides, the fictitious bodies of non-euclidean geometry do not exist, and cannot be accessible to experiment. Helmholtz, moreover, never explained himself altogether clearly on this point.

Lie pushed the analysis much farther. He inquired in what way the various possible movements of any system, or more generally the various possible transformations of a figure, can be combined. If we consider a certain number of transformations, and suppose that they are combined in all possible ways, the totality of all these combinations will form what he calls *a group*. To each group corresponds a geometry, and ours, which corresponds to the group of displacements of a rigid body, is only a very special case. But all the groups which one can imagine will possess certain common properties, and it is precisely these common properties which limit the caprice of the inventors of geometries; it is they, indeed, which Lie studied all his life.

He was, however, not entirely satisfied with his work. He had, he said, always regarded space as a *Zahlenmannigfaltigkeit*. He had confined himself to the study of continuous groups [271] properly so called, to which the rules of the ordinary infinitesimal analysis apply. Was he not thus artificially restricted? Had he not thus neglected one of the indispensable axioms of geometry (referring to the axiom of Archimedes)? I do not know whether any trace of this thought would be found in his printed works, but in his correspondence, or in his conversation, he constantly expressed this same concern.

This is precisely the gap which Professor Hilbert has filled up; the geometries of Lie remained all subject to the forms of analysis and of arithmetic, which seemed unassailable. Professor Hilbert has broken through these forms, or, if you prefer, he has enlarged them. His spaces are no longer *Zahlenmannigfaltigkeiten*.

The objects which he calls points, straight lines, or planes become thus purely logical entities which it is impossible to represent to ourselves. We should not know how to picture them as sensory images, these points which are nothing but systems of three series. It matters little to him; it is sufficient for him that they are *individuals* and that he has positive rules for distinguishing these individuals one from another, for establishing arbitrarily between them relations of equality or of inequality, and for transforming them.

One other comment: the groups of transformations in Lie's sense appear to play only a secondary part. At least this is how it seems when we read the actual text of Professor Hilbert. But, if we should consider it more closely, we should see that each of his geometries is still the study of a group. His non-archimedean geometry is the study of a group which contains all the transformations of the euclidean group, corresponding to the various displacements of a rigid body, but which contains also other transformations capable of being combined with the first according to simple laws.

Lobachevsky and Riemann rejected the postulate of Euclid, but they preserved the metrical axioms; in the majority of his geometries, Professor Hilbert does the opposite. This amounts to placing in the first rank a group comprising the transformations of space by the homothetic transformation and by translation; and at the foundation of his non-pascalian geometry we meet an analogous group, comprising not only the homothetic transformation and the translations of ordinary space, but other analogous transformations which combine with the first according to simple laws [272].

Professor Hilbert seems rather to slur over these inter-relations; I do not know why. The logical point of view alone appears to interest him. Being given a sequence of propositions, he finds that all follow logically from the first. With the foundation of this first proposition, with its psychological origin, he does not concern himself. And even if we have, for example, three propositions A, B, C, and if it is logically possible, by starting with any one among them, to deduce the other two from it, it will be immaterial to him whether we regard A as an axiom, and derive B and C from it, or whether, on the contrary, we regard C as an axiom, and derive A and B from it. The axioms are postulated; we do not know where they come from; it is then as easy to postulate A as C.

His work is then incomplete; but this is not a criticism which I make against him. Incomplete one must indeed resign one's self to be. It is enough that he has made the philosophy of mathematics take a long step in advance, comparable to those which were due to Lobachevsky, to Riemann, to Helmholtz, and to Lie.

Since* the printing of the preceding lines, Professor Hilbert has published a new note on the same subject ("Ueber die Grundlagen der Geometrie," *Nachrichten der K. Gesellschaft der Wissenschaften zu Göttingen*, 1902, Heft 3). He seems to have made here an attempt to fill in the gaps which I have noticed above. Although this note is very concise, one sees clearly two thoughts running through it. In the first place he seeks to present the axioms of order emancipated from all dependence on projective geometry; he uses for this a theorem of Professor Jordan. Next, he reconnects the fundamental principles of geometry with the notion of a group. He comes nearer then to the point of view of Lie, but he makes an advance on the work of his predecessor, since he frees the theory of groups from all appeal to the principles of the differential calculus.

H. POINCARÉ.

ON LINEAR DIFFERENTIAL CONGRUENCES.

BY DR. SAUL EPSTEEN.

(Read before the American Mathematical Society, April 25, 1903.)

IN his note entitled "Sur des congruences différentielles lineaires," Guldberg† concludes that there exists for linear differential forms a theory which is analogous to the Galois field theory. Being unable to find anything on this subject beyond that written by Guldberg, it may be permitted me to correct him in some points and to give a brief résumé of some additional results.

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*[See footnote at the beginning of this translation. Since this postscript was written, still another article by Hilbert has appeared: "Ueber die Grundlagen der Geometrie," *Math. Annalen*, vol. 56 (1902), pp. 381–422. *Tr.*]

†Guldberg, *Comptes rendus*, vol. 125 (1897), p. 489.

I define a differential Guldberg field ($DGuF$) analogous to the Galois field (GF) and show, 1° that every finite differential field is a $DGuF$, and 2° that there exists one and only one $DGuF$ of order p^n, p being a prime and n any integer.

§ 1.

This section is mainly a summary of Guldberg's results. We consider linear differential forms with integral coefficients

$$D(y) = Dy = \sum_{i=0}^{n} a_i \frac{d^i y}{dx^i}$$

and agree to understand the word "product" in the well known symbolic sense of Boole.*

If $D_1 y = D_2 y + p \cdot D_3 y$, then we may write the congruence $D_1 y \equiv D_2 y \pmod{p}$. When a differential expression Dy is given, we first reject the terms whose coefficients are multiples of p, then if $Dy = D_1 y \cdot D_2 y \pmod{p}$, we say that $D_1 y$ and $D_2 y$ are divisors of Dy modulo p.

In one and only in one of the *associated* forms $1\,D$, $2\,D$, $3\,D$, $\cdots$, $(p-1)D$ is the coefficient of the term of highest order congruent to 1 modulo p. This one is called the *principal* form.

A form D which is not divisible by other differential forms (except by its associated forms) is said to be *irreducible* or *prime* modulo p.

An algorithm analogous to Euclid's for finding the greatest common divisor holds† and from this it can be shown that: Any Dy (or simply D) can be decomposed in one and only one way into the product of an integer and irreducible principal forms.

The fact that D is divisible by Δ, modulo p can be expressed thus: $D \equiv 0 \pmod{p, \Delta}$, which means that $D = \phi \cdot \Delta + p \cdot D_1$. Likewise $D_1 \equiv D_2 \pmod{p, \Delta}$ means that $D_1 = \phi \cdot \Delta + pD_3 + D_2$. If Δ is of order n, any form D is congruent to one and only one of the p^n forms

$$\sum_{0}^{n-1} a_i \frac{d^i y}{dx^i}$$

* Boole's Differential Equations, p. 381.

† Although Guldberg's conclusions are correct in a general sort of way his notation is not always so, he does not seem to notice that in this theory integers cannot appear by themselves, but are always accompanied by y or a derivative of y (zero excepted). See also the correction at the end of this section.

moduli (p, Δ), where the coefficients a_i take all the values $0, 1, 2, \cdots, (p-1)$. These p^n expressions constitute a complete system of residues moduli (p, Δ).

The chief result attained by Guldberg is a generalization of Fermat's theorem: Let Δy be an irreducible form of order n and Dy a residue prime to Δ, then $Dy^{p^n-1} \equiv 1 \pmod{p, \Delta}$.*

§ 2. *The Differential Guldberg Field.*†

Let Δ [*i. e.*, $\Delta(y)$] be a linear homogeneous differential form of order n having integral coefficients not all divisible by a given integer p. If an arbitrary form D, likewise with integral coefficients, is divided by Δ there results a quotient Q and a remainder which can be written $\phi(y) + p \cdot \psi(y)$, where ϕ is of the form

$$\phi = \sum_{i=0}^{n-1} a_i \frac{d^i y}{dx^i},$$

each a_i belonging to the series $0, 1, 2, \cdots, (p-1)$. Then $D = \phi + p \cdot \psi + \Delta \cdot Q$. The totality of functions obtained by giving to Q and ψ all possible forms will be said to constitute a class of residues; two functions are congruent if, and only if, they belong to the same class of residues. Evidently there are p^n residues. From

$$D_i = \phi_i + p \cdot \psi_i + \Delta \cdot Q_i \qquad (i = 1, 2)$$

we see that the classes to which $D_1 \pm D_2$ or $D_1 \cdot D_2$ depends merely upon $\phi_1 \pm \phi_2$ and $\phi_1 \cdot \phi_2$. Suppose we differentiate D_1 twice and D_2 once and add

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* We shall see in § 2, No. 6, that this should read $Dy^{p^n-1} \equiv y \pmod{p, \Delta}$.

† In § 2, Dickson's Linear Groups, Chapters I and II, are closely followed. The analogy with the Galois field theory is necessarily very close, indeed at most times it is scarcely more than a difference in notation, differential forms taking the place of algebraic forms; for example:

In Galois field theory

$$(a_\alpha Z^\alpha + \cdots + a_0)(b_\beta Z^\beta + \cdots + b_0) = a_\alpha b_\beta Z^{\alpha+\beta} + \cdots + a_0 b_0.$$

In Guldberg field theory

$$\left(a_\alpha \frac{d^\alpha}{dx^\alpha} + \cdots + a_0\right)\left(b_\beta \frac{d^\beta}{dx^\beta} + \cdots + b_0\right) y = a_\alpha b_\beta \frac{d^{\alpha+\beta} y}{dx^{\alpha+\beta}} + \quad + a_\alpha b_3 y.$$

$$D_1'' + D_2' = \phi_1'' + \phi_2' + p(\psi_1'' + \psi_2') + \Delta'' Q_1 + \Delta'(2Q_1' + Q_2) + \Delta(Q_1'' + Q_2').$$

Since multiplication is symbolic we have $\Delta = \Delta \cdot y', \Delta'' = \Delta \cdot y''$, whence

$$D_1'' + D_1' = \phi_1'' + \phi_2' + p(\psi_1'' + \psi_2') + \Delta(y'' Q_1 + 2y' Q_1' + y_1' Q_2 + Q_1'' + Q_2').$$

Similarly, upon multiplying, there results an expression of the form $D_1'' \cdot D_2' = \phi_1'' \phi_2' + p \cdot \Phi + \Delta \cdot \Psi$; and in general, when we differentiate D_1 r times and D_2 s times, the results of addition and multiplication are of the form

$$D_1^{(r)} \pm D_2^{(s)} = \Phi_1 + p\Psi_1 + \Delta\theta_1$$
$$D_1^{(r)} \cdot D_2^{(s)} = \Phi_2 + p\Psi_2 + \Delta\theta_2.$$

Hence, after any number of differentiations, classes of residues combine without ambiguity, under addition, subtraction and multiplication. In order that division of any arbitrary class by any class $C(\neq 0)$ shall lead uniquely to a third class, it is necessary and sufficient that p shall be prime and Δ irreducible modulo p.

The p^n classes of residues above defined form what we shall call a Guldberg field $GuF(p^n)$ of order p^n (it being understood that p is a prime and Δ is irreducible modulo p). Sometimes for clearness we shall speak of it as a *differential* field $DGuF(p^n)$.

Example:

$$\text{Let } p = 3, \text{ and } \Delta = \frac{d^2y}{dx^2} - \frac{dy}{dx} - y.$$

The 3^2 residues are

$$0, y\left(=\frac{d^0y}{dx^0}\right), \quad -y, \quad \frac{dy}{dx}, \quad \frac{dy}{dx} + y, \quad \frac{dy}{dx} - y, \quad -\frac{dy}{dx},$$
$$-\frac{dy}{dx} + y, \quad -\frac{dy}{dx} - y.$$

We should notice at this point that there does not arise a numerical residue (excepting 0).

The sum, difference, product or derivative of any two of these may be reduced moduli 3 and $y'' - y' - y$ to one of the nine residues. Moreover the quotient of any one by any residue except 0 may be reduced to one of the set. It will be necessary, however, to bear in mind that multiplication and division are symbolic, for this reason it is best to write

$$y = \frac{d^0 y}{dx^0} = y^{(0)},$$

thus making it clear that

$$\frac{d^i y / dx^i}{y} = \frac{d^i y}{dx^i}.$$

$$\frac{y}{dy/dx} = \frac{y^{(0)}}{y'} = \frac{y^{(0)}(y' - y^{(0)})}{y'(y' - y^{(0)})} = \frac{y' - y^{(0)}}{y'' - y'} = \frac{y' - y^{(0)}}{y^{(0)}}$$

$$= y' - y^{(0)} = y' - y.$$

The nine residues thus form a $DGuF(3^2)$. It should be noticed that for purposes of multiplication $y = y^{(0)} = \frac{d^0 y}{dx^0}$ acts like an ordinary unit.

1° Theorem: If two differential forms D and Δ having integral coefficients admit of no common divisor containing y or its derivatives modulo p, p being a prime, we can determine two functions D_1 and Δ_1 having integral coefficients such that

$$D_1 \cdot D - \Delta_1 \cdot \Delta \equiv y \pmod{p}.$$

The proof is analogous to that in Linear Groups, page 8.

Consider a general field $F(s)$ composed of a finite number of elements $v_0, v_1, \cdots, v_{s-1}$ which have the property of *not vanishing when differentiated any number of times with respect to* x. As there exists every difference, the zero element must be a mark of the field. However, this zero element is in a sense extraneous, no other number can occur in the field unless multiplied by a v. This assumption is made in order to have all the differential forms which will arise homogeneous. Any number of differentiations of a mark of the field is still a mark of the field. We will therefore change the notation somewhat and write for the marks

$$o, u_0, u_1, \cdots, u_{s-2}.$$

Denoting by p the least integer such that $v_p = 0$, the p marks

$$o, u_0, u_1, \cdots, u_{p-2}$$

are all distinct.

2° Theorem: This integer p is a prime. (For the proof *cf.* Linear Groups, page 9.)

3° Theorem: The order of $F(s)$ is a power of p (*cf.* Linear Groups, page 10).

4° Theorem. Any mark u of $F(s=p^n)$ satisfies a differential equation of order $k \leqq n$, viz.,

$$\sum_{i=0}^{k} c_i \frac{d^i U}{dx^i} = 0 \qquad (c_k \neq 0,\ k \leqq n)$$

(*cf.* Linear Groups, page 10).

The least positive integer e for which

$$\frac{d^e u}{dx^e} = u = \frac{d^0 u}{dx^0}$$

we will call the *period* of the mark u and u will be said to belong to the exponent e. The marks $u^{(0)}, u', u'', \cdots, u^{(e-1)}$ are all distinct.

5° Theorem: The period of any mark ($\neq 0$) of the differential field $DF(p^n)$ is a divisor of $p^n - 1$ (*cf.* Linear Groups page 11).

6° Theorem: Every mark of the $DF(p^n)$ satisfies the equation

$$\frac{d^{p^n-1}u}{dx^{p^n-1}} = u.*$$

Since $d^e u/dx^e = u^{(0)}$ and $u^{(0)}, u', u'', \cdots, u^{(e-1)}$ are all distinct, it follows that all these e marks can be obtained from any one of them by differentiation; and it is clear then that when a certain number of marks $u_1, u_2, \cdots$ are given all the other marks of the field can be obtained from these by differentiation.

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* Guldberg (*loc. cit.*) incorrectly gives this as $\frac{d^{p^n-1}u}{dx^{p^n-1}} = 1$.

7° Theorem: If two marks u_1, u_2 belong respectively to the exponents e_1, e_2 which are relatively prime, their product $u_1 \cdot u_2$ belongs to the exponent $e_1 \cdot e_2$ and

$$\frac{d^{f_1}u_1}{dx^{f_1}} \quad \frac{d^{f_2}u_2}{dx^{f_2}} \quad \begin{pmatrix} f_1 = 0, 1, \cdots, e_1 - 1 \\ f_2 = 0, 1, \cdots, e_2 - 1 \end{pmatrix}$$

are all distinct (*cf.* Linear Groups, pages 11–12).

8° Theorem: Any equation of the kth order cannot have more than k linearly independent integrals (solutions) which belong to the field unless it is an identity, when every mark of the field is an integral.

9° Theorem: For every divisor f of $s-1$ the equation $d^f U/dx^f = U$ has in the DF $(s = p^n)$ exactly f solutions (*cf.* Linear Groups, page 12).

A mark belonging to the exponent $s-1$ is called a *primitive integral* of the equation $d^{s-1}U/dx^{s-1} = U$ and also a primitive integral of the DF (s). Since $\frac{du}{dx}, \frac{d^2u}{dx^2}, \cdots, \frac{d^{s-1}u}{dx^{s-1}}$ are all distinct we have

10° Theorem: The $p^n - 1$ marks $(\neq 0)$ of the DF $(s = p^n)$ are the $p^n - 1$ successive derivatives of a primitive integral of that field (*cf.* Linear Groups, page 13).

Corollary: If d is a divisor of $p^n - 1$, the mark $d^{\frac{p^n-1}{d}} u/dx^{\frac{p^n-1}{d}}$ belongs to the exponent d.

We may now identify the field $F(s)$ with the Guldberg field of order $s = p^n$ and show that not more than one such field can exist, in a manner quite analogous to the corresponding investigation for the Galois field theory (*cf.* Linear Groups, pages 13–14) and thus arrive at the theorem which corresponds to the theorem of Moore.*

Fundamental Theorem: *Every existent differential field of order s may be represented as a Guldberg field of order $s = p^n$. The $DGuF$ (p^n) is defined uniquely by its order; in particular, it is independent of the special irreducible differential congruence used in its construction.*

There remains now to be shown that *for any integer n there exists at least one irreducible differential congruence of order n.* This is easily accomplished in a manner which is analogous to

*E. H. Moore, Chicago Congress Mathematical Papers; BULLETIN, December, 1893.

that followed by Jordan in his Traité des Substitutions, pages 13–14.

A Galois field may be considered as made up of roots of unity $1, \epsilon, \epsilon^2, \cdots$ or what comes to the same thing, of a primitive root of unity and its various powers. A Guldberg field can be thought of as made up of e^x, $e^{\epsilon x}$, $e^{\epsilon^2 x}$, $\cdots$ (ϵ = root of unity) and the various derivatives of these quantities. From a certain point of view this field will be infinite since x is an independent variable.

THE UNIVERSITY OF CHICAGO,
April, 1903.

FIELDS WHOSE ELEMENTS ARE LINEAR DIFFERENTIAL EXPRESSIONS.

BY PROFESSOR L. E. DICKSON.

(Read before the American Mathematical Society, April 25, 1903.)

A. GULDBERG * has considered expressions of the form

$$Ay = a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1 \frac{dy}{dx} + a_0 y,$$

in which $a_n, \cdots, a_0$ are integers taken modulo p, p being a prime number. The *product* $Ay \cdot By$ of two such expressions in defined by Boole's symbolic method † to be

$$\left(a_n \frac{d^n}{dx^n} + \cdots + a_1 \frac{d}{dx} + a_0\right)\left(b_\beta \frac{d^\beta}{dx^\beta} + \cdots + b_1 \frac{d}{dx} + b_0\right) y,$$

so that the expansion may be effected as if d/dx were a constant. If, in this manner, $Ay \cdot By \equiv Cy \pmod p$, we say that Ay and By are *divisors* modulo p of Cy. Euclid's algorithm for the greatest common divisor is seen to hold. We may therefore define reducible and irreducible differential expressions modulo p. Let

$$\Delta y = \delta_n \frac{d^n y}{dx^n} + \cdots + \delta_1 \frac{dy}{dx} + \delta_0 y$$

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* "Sur des congruences différentielles linéaires," *Comptes rendus*, vol. 125, p. 489 (1897).

† Boole, Differential Equations, p. 381, seq.

be irreducible modulo p. Then any expression Ay is congruent modulis p, Δy with one and but one of the p^n expressions

$$c_{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \cdots + c_1\frac{dy}{dx} + c_0 y, \tag{1}$$

where $c_{n-1}, \cdots, c_0$ take independently the values $0, 1, \cdots, p-1$. Guldberg then states without proof certain theorems on congruences (modd p, Δy) with coefficients of type (1). The proofs of these and related theorems have recently been made by Dr. S. Epsteen, using the methods in the author's Linear Groups.

These results, however, follow at once from the Galois field theory if we show that the p^n differential expressions (1) define a field identical, aside from notation, with the $GF[p^n]$. To the expression (1) we make correspond the mark

$$c_{n-1}z^{n-1} + \cdots + c_1 z + c_0 \tag{2}$$

of the $GF[p^n]$, where z is a root of the congruence

$$\Delta' z = \delta_n z^n + \cdots \delta_1 z + \delta_0 \equiv 0 \pmod{p},$$

irreducible in view of the above assumption as to Δy. Since product relations are preserved by this correspondence (in view of the symbolic method for products), evidently any rational function of the differentials (1) corresponds to the same rational function of the corresponding marks (2).

An interesting feature of the field of the elements (1) is that the derivative of any element is again an element of the set.

An evident generalization consists in taking as the coefficients of the differential expressions elements of an arbitrary field instead of the special field of the integers modulo p.

THE UNIVERSITY OF CHICAGO,
April 20, 1903.

ON DIRECTRIX CURVES OF QUINTIC SCROLLS.

BY MR. CHARLES H. SISAM.

(Read before the American Mathematical Society, April 25, 1903.)

In various papers on quintic scrolls, particular forms of the surface have been found to contain a conic. The results are all included in the following theorem:

Every unicursal quintic scroll has three coplanar generators.

Let P_1, P_2 be two points of the nodal curve; let π_1, π_2 be the planes passing through P_1, P_2 respectively and containing the two generators which pass through those points.

The residual curve in each plane will be a nodal cubic. These two cubics may be expressed thus:

$$(1)\quad x = a\lambda^3 + b\lambda^2 + c\lambda, \quad y = a'\lambda^3 + b'\lambda^2 + c'\lambda + d',$$
$$z = a''\lambda^3 + b''\lambda^2 + c''\lambda, \quad w = 0.$$

$$(2)\quad x = a_1\lambda^3 + \cdots, \quad y = a_1'\lambda^3 + \cdots, \quad z = 0, \quad w = a_1\lambda^3 + \cdots.$$

The point $\lambda = 0$ is on both curves.

The generators join the point λ in (1) to the point λ in (2). If then three values λ can be found such that the six corresponding points are coplanar, the theorem will be established.

The line

$$(3)\qquad w = 0, \quad lx + my + nz = 0$$

lies in the plane π_1 and intersects the line

$$(4)\qquad z = 0, \quad lx + my + n'w = 0$$

which lies in the plane π_2.

The points in which the line (3) cuts the cubic (1) are defined by the values of λ which satisfy the equation

$$(la + ma' + na'')\lambda^3 + (lb + mb' + nb'')\lambda^2 + (lc + mc' + nc'')\lambda + md' = 0$$

and a similar cubic defines the points in which the line (4) cuts the cubic (2). If the values of l, m, n, n' can be so chosen

that the roots of these equations are respectively equal, then three generators intersect both the lines (3) and (4) and therefore lie in the plane determined by them.

It is no loss of generality to put $d' = d_1'$ and $n' = 1$. The conditions for the equality of the roots are now expressed by three linear equations in l, m, n. The solution is in general unique. The residual curve in the plane of the three generators is then a conic. There may be an infinite number of solutions. In this case the planes so determined all pass through a line which is a double directrix of the surface. Again, the scroll may be of such a form that every plane through two generators contains two other generators. The residual curve is then a rectilinear directrix. Hence:

*Every unicursal quintic scroll contains either a simple conic, a double directrix line or a simple rectilinear directrix. In each case the curve is unique and constitutes the simplest directrix curve on the surface.**

If a real quintic scroll has a four-fold line and a simple rectilinear directrix, the asymptotic lines are rational quartic curves when a certain condition is satisfied. *In this case, through every real point of the four-fold line — except the pinch-points — pass two real and two imaginary generators.*

Let the equations of the scroll be

$$z\phi(x, y) + w\phi_1(x, y) = 0.$$

in which

$$\phi(x, y) \equiv ax^4 + 4bx^3y + 6cx^2y^2 + 4dxy^3 + ey^4,$$

and

$$\phi_1(x, y) \equiv a_1x^4 + \cdots;$$

let $x/y = \lambda$, $w/z = \mu$. The equation becomes

$$\phi(\lambda) + \mu\phi_1(\lambda) = 0. \tag{5}$$

The equation of the asymptotic lines is

$$\frac{d\mu}{d\lambda} = k^2\frac{y^2}{z^2}.$$

* Cf. Snyder, "On the quintic scroll having three double conics," BULLETIN, vol. 9, p. 236.

The necessary and sufficient condition that the equation of the asymptotic lines be factorable is that $d\mu/d\lambda$ be a square, or that three unequal values of μ, as μ_1, μ_2, μ_3, exist for each of which (5) will have a cubic factor. The values μ are the roots of the equation

$$g_2 = 0.$$

Since $\mu_1 \neq \mu_2 \neq \mu_3$ and $g_2 = 0$ is a quadratic in μ, hence $g_2 \equiv 0$ and $\Delta = g_2^3 - 27g_3^2 = -27g_3^2$ is negative for all real values of μ except μ_1, μ_2, μ_3; hence two of the roots λ in (5) are real and two are imaginary except at the pinch points. But the generators through the point $(0, 0, 1, w)$ lie in the planes $x = \lambda y$, $w = \mu z$, hence two of the generators are real and two imaginary.

Conversely, it can be shown that if $g_2 \equiv 0$, the equation of the asymptotic lines is factorable, hence $g_2 \equiv 0$ is the necessary and sufficient condition that the asymptotic lines are reducible.

CORNELL UNIVERSITY,
March 27, 1903.

JOSIAH WILLARD GIBBS, PH.D., LL.D. A SHORT SKETCH AND APPRECIATION OF HIS WORK IN PURE MATHEMATICS.

JOSIAH WILLARD GIBBS, Professor of Mathematical Physics in Yale University, died after a short illness, April 28, 1903. The loss to pure science in America occasioned by his death is difficult to overestimate, and it seems peculiarly fitting that some record should be made in this BULLETIN of the work in pure mathematics of so illustrious a member of the AMERICAN MATHEMATICAL SOCIETY. Moreover, his publications have appeared in a form lacking somewhat in perspicuity and detail, and are rather inaccessible.

No reference will be made in the following to the achievement of Professor Gibbs in his chosen field of Thermodynamics. A complete account of his work in Physics will be found in the *American Journal of Science*, September, 1903.* It is proposed here to give a brief appreciation of the ideas underlying the following papers, which constitute his published work in pure mathematics:

*Some paragraphs of the present sketch have already appeared in this article.

"Elements of Vector Analysis," arranged for the use of Students in Physics. New Haven. 8°. Pp. 1–36 in 1881, and pp. 37–83 in 1884.

"On Multiple Algebra." Vice-Presidential Address before Section A of the American Association for the Advancement of Science, published in the *Proceedings* of the Association, vol. 33, 1886, pp. 37–66.

"On the Rôle of Quaternions in the Algebra of Vectors." *Nature*, vol. 43, 1891, pp. 511–514.

"Quaternions and the Ausdehnungslehre." *Nature*, vol. 44, 1891, pp. 79–82.

"Quaternions and Vector Analysis." *Nature*, vol. 48, 1893, pp. 364–367.

It is quite apparent from these titles that the interest of Professor Gibbs was most keen in the subject of Multiple Algebra. In fact, for several years he delivered at Yale University a course of lectures on this topic, to which reference will be made below.

The theory of dyadics as developed in the Vector Analysis of 1884 must be regarded as the most important of these contributions. For the Vector Analysis as an *algebra* does not fulfill the definition of the linear associative algebras of Benjamin Peirce, since the scalar product of vectors lies outside the vector domain, nor is it a geometric analysis in the sense of Grassmann, the vector product satisfying the combinatorial law but yielding a vector instead of a magnitude of the second order. While these departures from the systems mentioned testify to the great ingenuity and originality of the author and do not impair the utility of the system as a tool for the use of students in physics, they nevertheless expose the discipline to the criticism of the pure algebraist. Such objection falls to the ground however in the case of the theory of dyadics, for this yields for $n = 3$ a linear associative algebra of nine units, namely nonions, the general nonion satisfying an identical equation of the third degree, the Hamilton-Cayley equation (Cf. Vector Analysis, Wilson, Chapter V).

It is easy to make clear the precise point of view adopted by Professor Gibbs in this matter. This is well expounded in the Vice-Presidential Address, and also in his warm defence of Grassmann's priority rights as against Hamilton, in the first article in *Nature*. He points out that the key

to all matricular algebras is to be found in the open (or indeterminate) product, (*i. e.*, products in which no equations subsist between the factors), and, after calling attention to the brief development of this product in Grassmann's work of 1844, affirms that Sylvester's assignment of the date 1858 to the "second birth of algebra" (this being the year of Cayley's Memoir on Matrices), must be changed to 1844. Grassmann, however, ascribes very little importance to the open product, regarding it as affording no important applications. On the contrary, Professor Gibbs assigns to it the very first place in the three kinds of multiplication considered in the Ausdehnungslehre, since from it may be derived the algebraic and the combinatorial product, and shows in fact that both of these may be expressed in terms of indeterminate products. Thus the multiplication rejected by Grassmann becomes, from the standpoint of Professor Gibbs, the key to all others. The originality of the latter's treatment of the algebra of dyadics, as contrasted with the methods of other authors in the allied theory of matrices, consists exactly in this, that Professor Gibbs regards a matrix of order n as a multiple quantity in n^2 units, each of which is an indeterminate product of two factors. On the other hand, C. S. Peirce, who was the first to recognize (1870) the quadrate linear associative algebras identical with matrices, uses for the units a *letter pair*, but does not regard this combination as a product. In addition, Professor Gibbs, following the spirit of Grassmann's system, does not confine himself to *one* kind of multiplication of dyadics, as do Hamilton and Peirce, but considers two sorts, both originating with Grassmann. Thus it may be said that quadrate or matricular algebras are brought entirely within the wonderful system expounded by Grassmann in 1844.

As already remarked, the exposition of the theory of dyadics given in the Vector Analysis is not in accord with Grassmann's system. In a footnote of his Vice-Presidential Address Professor Gibbs shows the slight modification necessary for this purpose, while the subject has been treated in detail and in all generality in his Lectures on Multiple Algebra.

In order to bring the algebra of dyadics within the system invented by Grassmann it is only necessary to replace the dyadic of Vector Analysis, that is, the indeterminate product of two vectors, by the indeterminate product of two magnitudes of Grassmann's system the sum of whose orders (in general

different) is always equal to n, the order of the system itself.

The details are briefly as follows: Suppose that

$$e_1, e_2, \cdots, e_n$$

are the fundamental units of the system; then *derived* units are formed from these by taking all possible products according to the combinatorial law

$$e_i e_k = - e_k e_i.$$

Thus we obtain

$n(n-1)/1\cdot 2$ units of the second order, of the type $e_i e_k$;
$n(n-1)(n-2)/1\cdot 2\cdot 3$ of the third order, of the type $e_i e_k e_l$;
etc.,
n of the $(n-1)$th order, of the type $e_1 e_2 \cdots e_{n-1}$;

and one of the nth order, $e_1 e_2 e_3 \cdots e_n$, which for brevity we set equal to unity. The n units of order $n-1$ can be chosen and designated

$$\bar{e}_1, \bar{e}_2, \cdots, \bar{e}_n$$

such that $\bar{e}_j e_j = 1$, but $\bar{e}_i e_k = 0$, $i \neq k$.

From these units we form the fundamental dyads, by multiplying any unit of order r into any other of order $n-r$, using the sign / to indicate the indeterminate product. This gives for example,

n^2 fundamental dyads $e_i/\bar{e}_k$,
$[n(n-1)/1\cdot 2]^2$ " " $e_i e_j/\bar{e}_k \bar{e}_l$,

since, by Grassmann's regressive law of multiplication any unit of order $n-2$ may be expressed in the form $\bar{e}_k \bar{e}_l$.

From the fundamental dyads are formed as in the Vector Analysis the *dyadics*

$$\Phi_1 = \sum_{i,k} a_{i,k} e_i/\bar{e}_k, \qquad \Phi_2 = \sum_{i,j,k,l} a_{ij,kl} e_i e_j/\bar{e}_k \bar{e}_l, \qquad \text{etc.},$$

the a's being ordinary scalars.

Professor Gibbs next asks the question: What distributive products of dyads or dyadics are simple and useful? Of these,

he finds two, corresponding precisely to the *double dot* and *double cross products* of Dr. Wilson's Vector Analysis, page 306. For example, the first sort gives

$$e_i/\bar{e}_k : e_j/\bar{e}_l = 0 \text{ or } e_i/\bar{e}_l$$

according as $k \neq j$ or $k = j$.

This law of multiplication is exactly that of C. S. Peirce's vids, and accordingly the algebra of dyadics based upon the double-dot law of multiplication is precisely the matricular algebra of the author mentioned.

The *double-cross product* is defined by the law

$$e_i/\bar{e}_k \colon e_j/\bar{e}_l \equiv e_i e_j/\bar{e}_k \bar{e}_l,$$

i. e., gives a dyad of different type. In the Vector Analysis, however, no new type of dyad results, and we have, in contradiction to Grassmann's system for $n = 3$, one type of dyad only.

This second type of product is a distinct contribution to the algebra of dyadics, as the development on pages 306–331 of the Vector Analysis of Dr. Wilson abundantly proves.

For geometric application this comes out in the following very interesting way. Suppose $n = 4$; then quantities of the first order may be represented as points in ordinary space, and the general dyadic Φ_1 gives a general collineation T. Now the dyadic

$$\Phi_2 = \frac{\Phi_1 \colon \Phi_1}{1 \cdot 2}$$

is T expressed in line coördinates, *i. e.*, is the corresponding collineation of Plücker's line geometry; moreover

$$\Phi_3 = \frac{\Phi_1 \colon \Phi_1 \colon \Phi_1}{1 \cdot 2 \cdot 3}$$

is T expressed in tangential or plane coördinates, while Φ_4 is simply the determinant of T.

This brief sketch will show, therefore, that Professor Gibbs's exposition of the algebra of dyadics treats of these multiple quantities not only as linear operators on quantities of the first order, to which the investigations of all authors on the related

topic of matricular algebras are limited, but also *as operators within the entire derived domain of Grassmann.* The lectures contain a very simple and elegant treatment of various questions in the theory of matrices, including Sylvester's laws of nullity and latency, the proof of the identical equation and the reduction of a dyadic to a canonical form both in the general and special cases.

It is interesting and gratifying to note that the interest aroused in the general subject of Multiple Algebra among his pupils, stimulated and directed by Professor Pierpont through his lectures on Hypercomplex Numbers, has resulted in several noteworthy publications by the late Professor Starkweather and by Dr. Hawkes, which exhibit the method and spirit of Benjamin Peirce's memoir on Linear Associative Algebras in a new and remarkably fruitful light.

PERCEY F. SMITH.

SHEFFIELD SCIENTIFIC SCHOOL,
July, 1903.

NOTES.

AT the forty-second annual convention of the National educational association, held in Boston, July 6–10, 1903, a conference on mathematics was conducted by Professor D. E. SMITH. The topic considered was the organization and the work of associations of teachers of mathematics. Over two hundred teachers of mathematics from various parts of the country were present at the conference, and the following papers were read: "On the work of the Central association of science and mathematics teachers," by Mr. C. E. COMSTOCK; "On the New England Association," by Mr. E. H. NICHOLS; "On the proposed association for the middle states and Maryland," by Dr. J. S. FRENCH; "On the investigation being made in New England as to geometry in the grammar school," by Mr. W. T. CAMPBELL; "On the report of the committee appointed by the AMERICAN MATHEMATICAL SOCIETY, on college entrance requirements," by Professor H. W. TYLER; "The relation of these associations to the AMERICAN MATHEMATICAL SOCIETY," by Professor W. F. OSGOOD. From the inquiries made at the conference it is probable that several associations will be formed during the coming year.

At the meeting of the London mathematical society, held on June 11, the president, Professor H. Lamb, announced that after the conclusion of the present volume some changes would be made in the form of publication of the *Proceedings*, the chief being an increase in the size of the page and the type. The following papers were presented: "The application of quaternions to the algebra of invariants," by Major P. A. MacMahon; "Jacobi's construction for quadric surfaces," by Professor G. B. Mathews; "Automorphic functions in relation to the general theory of algebraic curves," by Mr. H. W. Richmond; "Addition to the paper on four known simple groups of order 25920," by Professor L. E. Dickson. An informal communication "On a method of introducing the logarithmic function by means of geometrical properties of conics" was made by Professor A. C. Dixon.

The Australasian association for the advancement of science will meet at Dunedin, New Zealand, in January next, under the presidency of Professor T. W. E. David, of the University of Sydney. Professor W. H. Bragg is president of Section A (astronony, mathematics, physics, and mechanics).

Oxford University. — The following courses in mathematics are announced for the Michaelmas term, 1903: — By Professor E. B. Elliott: Theory of numbers, two hours; Infinite series and products, one hour. — By Professor H. H. Turner: Elementary mathematical astronomy, two hours. — By Professor W. Esson: Analytic geometry of plane curves, two hours; synthetic geometry of plane curves, one hour. — By Professor A. E. H. Love: The mechanics of deformable bodies, two hours; Problems in applied mechanics, one hour. — By Mr. C. E. Haselfoot: Algebra, two hours. — By Mr. C. Leudesdorf: Projective geometry, three hours. — By Mr. A. E. Joliffe: Analytic geometry, two hours. — By Mr. J. W. Russell: Differential calculus, two hours. — By Mr. R. F. McNeile: Curve tracing, one hour. — By Mr. A. L. Pedder: Problems in pure mathematics, one hour. — By Mr. C. H. Sampson: Higher solid geometry, two hours. — By Mr. J. E. Campbell: Differential equations, two hours. — By Mr. C. H. Thompson: Integral calculus, two hours. — By Mr. E. H. Hayes: Analytic statics, three hours. — By Mr. A. L. Dixon: Hydrostatics, two hours. — By Mr. H. T. Gerrans: Advanced rigid dynamics, two hours. — By Mr. P. J. Kirby: Attractions and electrostatics, two hours.

THE German universities below announce for the winter semester, 1903–1904, courses in mathematics as follows:

UNIVERSITY OF GÖTTINGEN. — By Professor F. KLEIN: Differential and integral calculus, part 2, four hours; Seminar, theory of probabilities, two hours. — By Professor D. HILBERT: Partial differential equations, four hours; Number concept and quadrature of the circle, two hours; Seminar in differential equations, two hours. — By Professor H. MINKOWSKI: Mechanics, part 1, four hours; Geometry of numbers, two hours; Seminar (with Professor Hilbert) in differential equations, two hours. — By Professor M. BRENDEL: Geodesy, two hours; Mathematical statistic, one hour; Seminar in insurance, two hours. — By Professor F. SCHILLING: Descriptive and projective geometry, two hours, with exercises, four hours; Kinematics, one hour. — By Dr. E. ZERMELO: Calculus of variations, three hours; Determinants, one hour; Exercises in the calculus, one hour. — By Dr. O. BLUMENTHAL: Automorphic functions, two hours. — By Professors HILBERT and MINKOWSKI and Drs. ZERMELO and BLUMENTHAL: Algebraic-arithmetic exercises, one hour.

UNIVERSITY OF JENA. — By Professor J. THOMAE: Analytic geometry of space, four hours; elementary theory of functions, four hours; Seminar, two hours. — By Professor A. GUTZMER: Integral calculus, four hours; Determinants and algebra, four hours; Seminar exercises in integral calculus, one hour. — By Professor G. FREGE: Partial differential equations, four hours; Symbolic language, one hour.

THE Royal academy of sciences and letters of Denmark proposes the following prize question for the current year:

"To state the necessary and sufficient conditions for the decomposition of two polyhedra into a finite number of parts congruent two by two, or to make a contribution to the solution of this general problem by giving at least the conditions for the case where one of the solids is a convex polyhedron and the other a cube. There should also be expressly indicated what pyramids satisfy the conditions discovered."

The question is suggested by an address given by Professor Hilbert before the mathematical congress in Paris in 1901, and the subsequent researches of Messrs. Dehn and Vahlen (*Mathematische Annalen*, volumes 55, 56). The prize is the gold medal of the academy.

THE South African association for the advancement of science was inaugurated in April. The presidential address was delivered by Sir DAVID GILL, and the address before Section A (astronomy, chemistry, mathematics, meteorology and physics) by Professor P. D. HAHN.

PROFESSOR H. POINCARÉ has received the degree of doctor of science from Oxford University.

PROFESSOR MAX NOETHER of Erlangen has been elected a foreign member of the Hungarian academy of sciences at Buda Pesth.

PROFESSOR H. B. FINE, of Princeton University, has been made dean of the faculty.

PROFESSOR C. S. HOWE, of the Case School of Applied Science, Cleveland, Ohio, has been elected to the presidency of that institution.

PROFESSOR F. S. LUTHER, of the chair of mathematics and dean of the faculty of Trinity College, Hartford, Conn., has been made acting president of the college.

DR. E. R. HEDRICK, of Yale University, has been appointed to the professorship of mathematics in the University of Missouri.

AT the University of Michigan Dr. J. W. GLOVER has been promoted to an assistant professorship of mathematics; Dr. PETER FIELD and Dr. J. N. JAMES have been appointed instructors in mathematics.

DR. C. M. MASON has been appointed to an instructorship in mathematics in the Massachusetts Institute of Technology.

PROFESSOR LUIGI CREMONA died at Rome on June 10. Professor Cremona was born at Pavia, December 7, 1830. After holding the chairs of mathematics in the University of Bologna and the Polytechnic School of Milan, he was called in 1873 to the University and the directorship of the Engineering School at Rome, where the remainder of his life was spent. He was a member of the Italian Senate and had served as vice-president of the Senate and as minister of public instruction.

ERNEST DUPORCQ, one of the editors of the *Nouvelles Annales Mathématiques* and general secretary of the Paris congress of mathematicians, died on April 1, at the age of thirty years.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

BAUER (G.). Vorlesungen über Algebra. Herausgegeben vom mathematischen Verein München. Mit dem Bildnis Gustav Bauers als Titelbild. Leipzig, Teubner, 1903. 8vo. 384 pp. M. 10.00

BERIDZE (I. R.). See KORKIN (A. N.).

BOLYAI DE BOLYA (I.). Appendix scientiam spatii absolute veram exhibens: a veritate aut falsitate axiomatis XI Euclidei, a priori haud unquam decidenda, independentem: adiecta ad casum falsitatis quadratura circuli geometrica. Editio nova oblata ab Academia scientiarum hungarica ad diem natalem centesimum auctoris concelebrandum. Ediderunt I. Kürschák, M. Réthy, B. Törössy de Zepethnek. Leipzig, Teubner, 1903. 4to. 7 + 40 pp., 7 plates. M. 4.00

——. See LIBELLUS.

BONOLA (R.). See LIBELLUS.

BOY (W.). Ueber die Abbildung der projektiven Ebene auf eine im Endlichen geschlossene singularitätenfreie Fläche. Abhandlung zu den Modellen der Serie XXX, No. 1 und 2. Halle, Schilling, 1902. 8vo. 14 pp. (Mathematische Abhandlungen aus dem Verlage mathematischer Modelle von Martin Schilling in Halle a. S., neue Folge, No. 5.) M. 0.80

BURKHARDT (H.). Funktionentheoretische Vorlesungen. Vol. I, Heft 1: Algebraische Analysis. Leipzig, Veit, 1903. 8vo. 12 + 195 pp. M. 5.20

FAZZARI (G.). See MASCHERONI (L.).

FRANKENBACH (F. W.). Die den merkwürdigen Punkten des Dreiecks entsprechenden einbeschriebenen und umschriebenen Kegelschnitte. (Progr.) Liegnitz, 1903. 8vo. 44 pp.

FRANZ (K.). Ueber die hypergeometrische Differentialgleichung mit Nebenprodukten. (Progr.). Berlin, Weidmann, 1903. 4to. 20 pp. M. 1.00

GEIGENMÜLLER (R.). Leitfaden und Aufgabensammlung zur höheren Mathematik. Für technische Lehranstalten und den Selbstunterricht bearbeitet. Vol. II: Die höhere Analysis oder Differential- und Integralrechnung. 5te Auflage. Mittweida, Polytechnische Buchhandlung, 1903. 8vo. 15 + 334 pp. Cloth. M. 7.50

GLASER (S.). Untersuchung der Flächen dritten Grades, welche bei der Abbildung nach dem Prinzip der reziproken Radienvektoren wieder in sich selbst zurückkehren. Teil 2. (Progr.) Berlin, Weidmann, 1903. 4to. 31 pp. M. 1.00

HEDRICK (E. R.). On the characteristics of differential equations. 4to. (*Annals of Mathematics*, second series, Vol. 4, pp. 121–159).

HENNEBERGER (M.). Beiträge zur Theorie der Integrale der Bernoullischen Funktion. Bern, 1902. 8vo. 67 pp. M. 2.00

JAHNKE (E.). Nachruf auf Ferdinand Caspary. Mit dem Bildnis F. Casparys, einem Verzeichnis seiner Abhandlungen, sowie einem Briefe Ch. Hermites an H. Bertram. Leipzig, Teubner, 1903. 8vo. 30 pp. (From *Jahresbericht der Deutschen Mathematiker-Vereinigung*, Vol. 12.) M. 1.40

KEYSER (C. J.). The plane geometry of the point in point-space of four dimensions. 4to. (*American Journal of Mathematics*, Vol. 25, pp. 301–330).

——. Concerning the angles and the angular determination of planes in 4-space. 8vo. (*Bulletin of the American Mathematical Society*, (2) Vol. 8, pp. 324–329.)

KÖNIG (J.). Einleitung in die allgemeine Theorie der algebraischen Grössen. Aus dem Ungarischen übertragen vom Verfasser. Leipzig, Teubner, 1903. 8vo. 10 + 564 pp. M. 16.00

KORKIN (A. N.). Integration of differential equations and calculus of variations; lectures, edited by I. R. Beridze. St. Petersburg, 1903. 8vo. 375 pp. (Lith.) (Russian.)

KÜRSCHÁK (I.). See BOLYAI DE BOLYA (I.).

KWIETNIEWSKI (S.). Ueber Flächen des vierdimensionalen Raumes deren sämtliche Tangentialebenen untereinander gleichwinklig sind, und ihre Beziehung zu den ebenen Kurven. (Diss.) Zürich, Speidel, 1902. 8vo. 51 pp. M. 1.00

LIBELLUS post saeculum quam Ioannes Bolyai de Bolya anno MDCCCII a. d. XVIII Kalendas ianuarias Claudiopoli natus est, ad celebrandam memoriam eius immortalem ex consilio ordinis mathematicorum et naturae scrutatorum regiae litterarum universitatis hungaricae Francisco-Josephinae Claudiopolitanae editus. Claudiopoli, 1902. (Leipzig, Teubner.) 4to. 15 + 154 pp., 1 facsimile.

(Contents: Latin translation of J. Bolyai's letter to his father, Nov. 3, 1823; L. Schlesinger, Ueber einige funktionentheoretische Anwendungen der absoluten Geometrie; P. Stäckel, Ueber die Mechanik mehrfacher Mannigfaltigkeiten; R. Bonola, Bibliography of absolute geometry 1837–1902.)

MANNOURY (G.). Over de beteekenis der wiskundige logica voor de philosophie. Rotterdam, 1903. 8vo. 16 pp. M. 1.00

MASCHERONI (L.). La geometria del compasso. Nuova edizione, con prefazione e note da G. Fazzari. Palermo, 1902. 8vo. Fr. 2.00

MILHAUD (G.). Les philosophes-géomètres de la Grèce; Platon et ses prédécesseurs. Paris, 1900. 8vo. 392 pp. Fr. 6.50

MILLER (G. A.). A new proof of the generalized Wilson's theorem. (*Annals of Mathematics*, (2) 4, pp. 188–190.) 4to.

MORITZ (R. E.). Ueber Continuanten und gewisse ihrer Anwendungen im zahlentheoretischen Gebiete. (Diss.) Göttingen, Kästner, 1903. 8vo. 36 pp.

PALMIERI (F. S.). I gruppi di movimenti nelle metriche subordinate alla proiettiva. Parte I: Le forme di 2a specie. Roma, Cuggiani, 1902. 8vo. 143 pp.

PECH (R.). Ueber Modulargleichungen elliptischer Funktionen. (Progr.) Gross-Strehlitz, Wilpert, 1903. 4to. 10 pp. M. 1.00

REICHENBÄCKER (E.). Ueber Transformation unendlicher Reihen. (Diss.). Halle, 1903. 8vo. 63 pp.

RÉTHY (M.). See BOLYAI DE BOLYA (I.).

SCHLESINGER (L.). See LIBELLUS.

SCHMIDT (M.). Analogien in der analytischen Geometrie der Ebene und des Raumes. Kempten, 1903. 8vo. 50 pp., 9 plates. M. 2.00

SEYFERT (R.). Ueber die Auffassung einfachster Raumformen. (Diss.) Leipzig, 1902. 8vo. 28 pp., 1 plate.

STÄCKEL (P.). See LIBELLUS.

TÖRÖSSY DE ZEPETHNEK (B.). See BOLYAI DE BOLYA (I.).

WELTZIEN (K.). Die $n$te Wurzel aus einer linearen Substitution. (Progr.) Berlin, Weidmann, 1903. 4to. 22 pp. M. 1.00

ZACHARIAS (M.). Ueber die Beziehungen zwischen den 27 Geraden auf einer Fläche 3ter Ordnung und den 28 Doppeltangenten einer ebenen Kurve 4ter Ordnung. (Diss.) Göttingen (Berlin, Mayer & Müller), 1903. 8vo. 38 pp.

II. ELEMENTARY MATHEMATICS.

ALBA Y CLARES (L. DE). Ejercicios de la Geometría de M. Ortega, resueltos. Malaga, La Equitativa, 1903. 8vo. 175 pp. Fr. 4.50

BARRELL (F. R.). Elementary geometry. Section 1, part 2: subject matter of Euclid, book III. 1–34, and book IV. 4–9. London, Longmans, 1903. 8vo. Cloth. 1s.

——. Elementary geometry. Section 1, part 1: subject matter of Euclid, book 1. London, Longmans, 1903. 8vo. 128 pp. Cloth. 1s.

——. Elementary geometry. Section 1, parts 1 and 2: subject matter of Euclid, book I, book III. 1–34, and book IV. 4–9. London, Longmans, 1903. 8vo. Cloth. 2s.

BAUBEAU (M.). Premières notions d'algèbre (programme du 31 mai 1902) (enseignement secondaire, premier cycle, classe de cinquième B). Paris, Combet, 1903. 16mo. 104 pp.

BÖTTGER (A.). Die Stereometrie, für den Unterricht an der Realschule bearbeitet. 2te, vermehrte Auflage. Leipzig, Dürr, 1903. 8vo. 48 pp. Boards. M. 0.60

BRAUNE (A.). Raumlehre für Volks-, Bürger- und Fortbildungsschulen, sowie für Präparanden-Anstalten. Nach methodischen Grundsätzen bearbeitet. Bearbeitet von F. Skorczyk. 7te Auflage. Halle, Schroedel, 1903. 8vo. 5 + 84 pp. Boards. M. 0.75

BROOKSMITH (F. J.). See WOOLWICH.

BURG (R.). Sammlung algebraischer Aufgaben für gewerbliche und technische Lehranstalten, nebst einer Abhandlung über das Stab-

rechnen. Heft 3: Allgemeine Potenzen und Logarithmen; Gleichungen (2ter Teil); Verhältnisse und Proportionen (2ter Teil); vollständige quadratische Gleichungen. Frankfurt a. M., Auffarth, 1903. 8vo. 52 pp. M. 0.75

COMBETTE (E.). Précis d'algèbre élémentaire. Classes de seconde et de première C et D. Paris, Alcan, 1903. 8vo. Fr. 3.00

——. Cours abrégé d'arithmétique. Classes de mathématiques A et B. 7e édition revue et augmentée. Paris, Alcan, 1903. 8vo. Fr. 2.80

FUCINI (C.). Algebra elementare per gl'istituti nautici. Parte I. 2a edizione. Genova, Tipografia della Gioventù, 1903. 8vo. 7 + 134 pp. Fr. 1.80

GRÉVY (A.). Algèbre, à l'usage des élèves des classes de quatrième B à première C et D (programmes du 31 mai 1902). Paris, Nony, 1903. 18mo. 7 + 331 pp. Fr. 2.50

GUBLER (E.). Aufgaben aus der allgemeinen Arithmetik und Algebra für Mittelschulen. Heft 1. Zürich, Füssli, 8vo. 48 pp. Boards. M. 0.80

GUICHARD (C.). Traité de géométrie. Deuxième partie: compléments. Paris, Nony, 1903. 8vo. 8 + 430 pp.

HALBERSTADT (K.). Solutions of F. Bychkov's Problems in algebra. 2d enlarged edition. Part I. Vienna, 1902. 8vo. 384 pp. (Russian.)

HOLZMÜLLER (G.). Methodisches Lehrbuch der Elementarmathematik. Teil 3: Lehr- und Uebungsstoff zur freien Auswahl für die Oberklassen realistischer Vollanstalten und höherer Fachschulen, nebst Vorbereitungen auf die Hochschulmathematik. 2te Auflage, im Anschluss an die neuen preussischen Lehrpläne mit besonderer Berücksichtigung der Anwendungen bearbeitet. Leipzig, Teubner, 1903. 8vo. 14 + 370 pp. Cloth. M. 4.40

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——. See SCHRÖN (L.).

IKIN (A. E.). Knotty points in Euclid. Books 1 and 2. (Normal tutorial series.) London, Simkin, 1903. 8vo. Limp leather. 1s. 6d.

KÖNIGBAUER (J.). Geometrische Aufgaben für Mittelschulen und Lehrerbildungs-Anstalten. Resultate und Lösungsandeutungen. 4te Auflage. Regensburg, Habbel, 1903. 8vo. 56 pp. M. 0.60

KUTNEWSKY (M.). See MÜLLER (H.).

LA MANNA COPPOLA (G.). Lo sviluppo di un arco o del cerchio e la trisezione dell'angolo. Palermo, Vena, 1902. 4to. 20 pp., 4 plates.

LYDON (N. S.). A junior geometry. London, Methuen, 1903. 8vo. 180 pp. Cloth. 2s.

MARKS (C. I.). Mathematical questions and solutions from the *Educational Times*. New series. Vol. III. London, Hodgson, 1903. 8vo. Cloth. 6s. 6d.

MARSHALL (F.). See PEACHELL (F. H.).

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——. Heft 5: Quadratische Gleichungen mit einer und mehreren Unbekannten; höhere Gleichungen, welche sich auf quadratische zurückführen lassen. Heft 8: Die geometrischen Oerter Ellipse, Hyperbel, Parabel, mit vollständig gelösten Aufgaben. Freiburg i. B., Lorenz, 1903. 8vo. 43 + 39 pp. M. 2.00

MILLIS (C. T.). Technical arithmetic and geometry. For technical institutes, modern schools, workshops. London, Methuen, 1903. 8vo. 268 pp. Cloth. 3s. 6d.

MÜLLER (H.) und KUTNEWSKY (M.). Sammlung von Aufgaben aus der Arithmetik, Trigonometrie und Stereometrie. Ausgabe A, für Gymnasien und Progymnasien. Teil I. 2te, verbesserte Auflage. Leipzig, Teubner, 1903. 8vo. 7 + 237 pp. Cloth. M. 2.20

——. Ausgabe B, für reale Anstalten und Reformschulen. Teil I. 2te, verbesserte Auflage. Leipzig, Teubner, 1903. 8vo. 8 + 301 pp. Cloth. M. 2.80

OPGAVEN, wiskundige, met de oplossingen door de leden van het wiskundig genootschap. Nieuwe reeks. Deel VIII: 1899–1902. Amsterdam, 1903. 8vo. 5 pp. and pp. 265–400, 3 plates. M. 12.00

PEACHELL (F. H.) and MARSHALL (F.). Oxford and Cambridge theoretical and practical geometry. London, Gill, 1903. 8vo. Cloth. 1s. 6d.

PLEBANI (B.). Contro le pretese ciclometriche dell'ing. A. Foschini state glorificate da S. E. Guido Baccelli, Ministro dell'istruzione. Trattato di vera ciclometria, coll'aggiunta di un mesolario trigonometrico universale e di una trisettrice universale dell'angolo. Torino, Paravia, 1903. 8vo. 302 pp. Fr. 5.00

ROEDER (H.). Lehrsätze und Aufgaben aus der Planimetrie. 3te, verbesserte Auflage. Breslau, Hirt, 1903. 8vo. 102 pp. Boards. M. 1.00

ROHR (H.). Ein Beitrag zur sphärischen Trigonometrie. (Progr.) Breslau, 1903. 4to. 23 pp.

SCHRÖN (L.). Tables de logarithmes à sept décimales, pour les nombres 1 jusqu'à 108 000, et pour les lignes trigonométriques de dix secondes en dix secondes; et tables d'interpolation pour le calcul des parties proportionnelles; précédées d'une introduction par J. Houël. Paris, Gauthier-Villars, 1903. 8vo. Fr. 10.00

SKORCZYK (F.). See BRAUNE (A.).

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——. Lehrbuch der ebenen Geometrie mit Uebungsaufgaben, für höhere Lehranstalten. Ausgabe B: für mittlere Klassen. 9te, verbesserte Auflage. 30.–34. Tausend. Potsdam, Stein, 1903. 8vo. 5 + 172 pp. M. 1.60

STREMPEL. Neue, sehr genaue und einfache Methoden der Rektifikation sowie der Teilung von Kreisbogen. (Progr.) Rostock, 1903. 8vo. 28 pp., 2 plates.

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WARREN (A. T.). Experimental and theoretical course of geometry. London, Frowde, 1903. 8vo. 256 pp. Cloth. 2s.

WELLS (W.). A complete trigonometry, plane and spherical, with four-place tables. Boston, Heath, 1903. 148 pp. Half leather. $1.08

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III. APPLIED MATHEMATICS.

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ARRHENIUS (S.). Text-book of electrochemistry. Translated from the German edition by John McCrae. New York, Longmans, 1903. 8vo. 6 + 344 pp. $3.50

BECKER (A.). Kristalloptik. Eine ausführliche elementare Darstellung aller wesentlichen Erscheinungen, welche die Kristalle in der Optik darbieten, nebst einer historischen Entwickelung der Theorien des Lichts. Stuttgart, 1903. 8vo. 10 + 362 pp. M. 8.00

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BRESLAUER (M.). Das Kreisdiagramm des Drehstrommotors und seine Anwendung auf die Kaskadenschaltung. Stuttgart, Enke, 1903. 8vo. 4 + 90 pp. (Sammlung elektrotechnischer Vorträge.) M. 3.60

BROUGH (B. H.). Treatise on mine-surveying. 10th edition, revised. London, Griffin, 1903. 8vo. 388 pp., diagrams. Cloth. 7s. 6d.

BURZAGLI (V.). Le forze astratte nelle teorie fisiche. Modena, Forghieri e Pellequi, 1903. 8vo. 24 pp.

CAMERER (R.). Neue Diagramme zur Turbinentheorie. Berlin, 1902. 8vo. 30 pp. M. 1.00

COURTOIS (A. H.). Application du principe de la conservation de l'énergie au fonctionnement des ventilateurs centrifuges. Paris, 1903. 4to. 88 pp. Fr. 5.00

CRAPPER (E. H.). Electric and magnetic circuits. New York, Longmans, 1903. 8vo. 12 + 380 pp. $3.60

CROY (F.). Lehrbuch der niederen Geodäsie. Leipa, Künstner, 1903. 8vo. 24 + 728 pp., 3 plates. Half-mor. M. 18.00

DOYÈRE (C.). Notions élémentaires de mécanique du navire, ouvrage rédigé conformément aux programmes des examens. Première partie: examen théorique. Paris, Challamel, 1903. 8vo. 8 + 245 pp., 1 plate.

DUHEM (P.). Recherches sur l'hydrodynamique. (En 2 volumes.) 1re série: Principes fondamentaux de l'hydrodynamique; propagation des discontinuités, des ondes et des quasi-ondes. Paris, Gauthier-Villars, 1903. 4to. Fr. 10.00

ELOLA (J. DE). Planimetría de precisión ó estudios topográficos de análisis planimétrico. Obra premiada en 20 de diciembre de 1901 por la Escuela especial de Ingenieros de minas en concurso de premios costeados por el legado Gómez-Pardo, y tambien recompensada por el Ministerio de la Guerra, segú Reales órdenes de 23 de julio de 1902 y 8 de enero de 1903. Madrid, Sucesores de Rivadeneyra, 1903. 4to. Vol. I: 681 pp.; Vol. II: 714 pp.; Atlas of 52 plates. Fr. 50.00

EMCH (A.). Some special algebraic transformations realized by linkages. 8vo. (*University of Colorado studies*, Vol. 1, pp. 211–218.)

F. (F.). Abrégé du cours de géométrie appliquée au dessin linéaire. Paris, Poussielgue, 1903. 18mo. 112 pp.

FELDMANN (C.). See HERZOG (J.).

FOSTER (G. C.). See JOUBERT.

FOURNIER (E. F.). Recherches des lois dynamiques de l'air dans les cyclones et leur application à la détermination de la route de sécurité des navires dans ces ouragans. Paris, Chapelot, 1903. 8vo. 47 pp. Fr. 3.00

FUHRMANN (A.). Application of higher mathematics to questions in the physical sciences. Translation by B. Gushchin. St. Petersburg, 1903. 8vo. 504 pp. (Russian.) M. 10.00

GOULD (E. S.). Practical hydrostatics and hydrostatic formulas. (Van Nostrand scientific series, No. 117.) New York, Van Nostrand, 1903. 16mo. 114 pp. Cloth. $0.50

GRASSMANN (H.). Die Drehung eines kraftfreien starren Körpers um einen festen Punkt. Abhandlung zu den Modellen der Serie XXIX: Drei Modelle zur Kreiseltheorie. Halle, Schilling, 1903. 8vo.

51 pp. (Mathematische Abhandlungen aus dem Verlage mathematischer Modelle von Martin Schilling, in Halle a. S., neue Folge, No. 4.) M. 1.60

GROTENDURST (N. C.). Verzameling van vraagstukken ter oefening in de beginselen der mechanica. Breda, 1903. 8vo. 2 + 57 pp. M. 2.50

GRÜNWALD (F.). Die Herstellung und Verwendung der Accumulatoren in Theorie und Praxis. 3te, vermehrte und verbesserte Auflage. Halle, 1903. 12mo. 6 + 158 pp. Boards. M. 3.00

HELLAND-HANSEN (B.). See SANDSTRÖM (J. W.).

HERZOG (J.) und FELDMANN (C.). Die Berechnung elektrischer Leitungsnetze in Theorie und Praxis. 2te Auflage, in 2 Teilen. Teil I: Strom- und Spannungsverteilung in Netzen. Berlin, Springer, 1903. 8vo. 9 + 402 pp. Cloth. M. 12.00

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